PHY 308/578 **QUANTUM PHYSICS** **Spring 2006**

Final Exam with Solutions

==

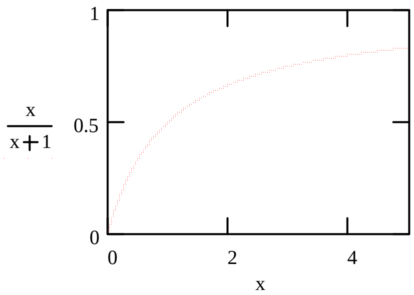
Problem F.1

A stream of 1D particles of mass *m* is incident on a very short and deep quantum well which may be described by potential energy $U(x) = -W\delta(x)$, with $W > 0$. Calculate and sketch the explicit dependence of the fraction of transmitted particles on their energy *E*.

Solution:

A similar problem for $U(x) = +W\delta(x)$ has been solved in class. Reviewing the solution, we see that the sign of *W* does not affect the final answer:

$$|T|^2 = \frac{1}{1+\left(\dfrac{mW}{\hbar^2 k}\right)^2} = \frac{E}{E+E_0}, \quad E = \frac{\hbar^2 k^2}{2m}, \quad E_0 \equiv \frac{mW^2}{2\hbar^2}$$



This function equals 0 at *E* = 0, and approaches 1 as *E* is increased well above the constant E_0 which characterizes the well – see the plot on the right.

Problem F.2

A 1D harmonic oscillator is initially in its ground state. Suddenly and rapidly, the oscillator's spring constant is increased so that its frequency doubles. Calculate the probability that the oscillator remains in the ground state.

Solution:

This problem is conceptually similar to homework Problem 3.1. The ground state wavefunction of the initial oscillator,

$$u_{ini}(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \exp\left(-\frac{m\omega x^2}{2\hbar}\right),$$

serves as an initial condition for the dynamics of the finite oscillator whose eigenfunctions can be obtained by the replacement $\omega \to 2\omega.$ In particular, the new ground state is

$$u_{fin}(x) = \left(\frac{2m\omega}{\pi\hbar}\right)^{1/4} \exp\left(-\frac{m\omega x^2}{\hbar}\right).$$

The probability p_0 we are looking for is just a modulus square of the expansion coefficient ("overlap integral")

$$A_0 = \int_{-\infty}^{+\infty} u_{ini}(x) u^*_{fin}(x)\,dx = 2^{1/4}\left(\frac{m\omega}{\pi\hbar}\right)^{1/2} \int_{-\infty}^{+\infty} \exp\left(-\alpha x^2\right)dx,$$

$$\alpha \equiv \frac{m\omega}{2\hbar} + \frac{m\omega}{\hbar} = \frac{3m\omega}{2\hbar},$$

so that introducing a new integration variable $\xi \equiv \sqrt{\alpha}\,x,$ we get

$$A_0 = 2^{1/4}\left(\frac{m\omega}{\pi\hbar\alpha}\right)^{1/2} \int_{-\infty}^{+\infty} \exp\left(-\xi^2\right)d\xi = 2^{1/4}\left(\frac{m\omega}{\hbar\alpha}\right)^{1/2} = 2^{1/4}\left(\frac{2}{3}\right)^{1/2},$$

so that finally $p_0 = |A_0|^2 = \sqrt{8/9} \approx 94\%.$

Problem F.3

A 3D quantum particle is confined to the interior of an infinitely long cylinder with a square-shaped cross-section: $0 \le x, y \le a$. Find particle's eigenfunctions and eigenenergies. What is the ground state energy?

Solution:

Inside the cylinder, we may select $U(\mathbf{r}) = 0$ and write the 3D Schrödinger equation as

$$-\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)u(x,y,z) = Eu(x,y,z).$$

The variable-separating solution, which satisfies both the equation and the boundary conditions ($u = 0$ at $x, y = 0, a$) has the form

$$u(x,y,z) = C\sin\frac{\pi n_x x}{a}\sin\frac{\pi n_y y}{a}\exp(ik_z z),$$

with n_x, n_y = 1, 2,…, and arbitrary (real) k_z. The corresponding eigenenergy is

$$E_{n_x,n_y}(k_z) = \frac{\hbar^2}{2m}\left(\frac{\pi^2}{a^2}n_x^2+\frac{\pi^2}{a^2}n_y^2+k_z^2\right).$$

Its minimum (ground-state) value is obviously $E_{\min} = \dfrac{\hbar^2}{m}\dfrac{\pi^2}{a^2}$.

Problem F.4

Calculate the average value and the r.m.s. deviation from the average of the angular momentum component L_x in an arbitrary eigenstate $\left|l,m\right\rangle$ of a spinless 3D particle in a central potential.

Solution: From the definition of operators $L_\pm$, we have

$$L_x = \frac{1}{2}(L_+ + L_-),\; L_x^2 = \frac{1}{4}(L_+^2 + L_-^2 + L_+L_- + L_-L_+).$$

Now we can use the expressions for $L_\pm\left|l,m\right\rangle$, derived in class, to find

$$\left\langle L_x\right\rangle = \left\langle l,m\right|L_x\left|l,m\right\rangle = \frac{1}{2}\Big(\left\langle l,m\right|L_+\left|l,m\right\rangle + \left\langle l,m\right|L_-\left|l,m\right\rangle\Big)$$

$$= \frac{1}{2}\Big(\left\langle l,m\right|const\left|l,m+1\right\rangle + \left\langle l,m\right|const\left|l,m-1\right\rangle\Big) = 0;$$

$$\left\langle L_x^2\right\rangle = \frac{1}{4}\begin{pmatrix}\left\langle l,m\right|const\left|l,m+2\right\rangle + \left\langle l,m\right|const\left|l,m-2\right\rangle \\ +\left\langle l,m\right|\hbar^2\left[l(l+1)-m(m-1)\right]\left|l,m\right\rangle + \left\langle l,m\right|\hbar^2\left[l(l+1)-m(m+1)\right]\left|l,m\right\rangle\end{pmatrix}$$

$$= \frac{\hbar^2}{2}\left[l(l+1)-m^2\right],$$

so that

$$\Delta L_x = \left(\left\langle L_x^2\right\rangle - \left\langle L_x\right\rangle^2\right)^{1/2} = \frac{\hbar}{\sqrt{2}}\left[l(l+1)-m^2\right]^{1/2}.$$

As could be expected from the (generally, naïve), classical picture of orbital motion, the uncertainty is smallest when the angular momentum vector is oriented along z-axis, e.g., $m = \pm l$:

$$\underline{\left(\Delta L_x\right)_{\min} = \hbar\left(l/2\right)^{1/2}.}$$

Problem F.5

Find the explicit form of spherical harmonics $Y_4^0(\theta,\varphi)$ and $Y_4^4(\theta,\varphi)$, besides the normalization coefficients. (The lower index stands for l, while the upper one is for *m*.)

Solution:

As has been discussed in class (see also the table on p. 127 of S.G.'s textbook), $Y_l^l(\theta,\varphi) = const \times \exp(il\varphi)\sin^l\theta,$ so that $Y_4^4(\theta,\varphi) = const \times \exp(4i\varphi)\sin^4\theta.$

Since $Y_l^m \propto \exp(im\varphi)\Theta(\theta),$ in the case $l = 4,\ m = 0,$ the azimuth-angle part is constant, and we only need to find $\Theta(\theta) \propto P_4^0(\cos\theta) = P_4(\cos\theta).$ In class, we have discussed the easiest formula which may be used for calculating the Legendre polynomials:

$$P_l(\xi) = \frac{1}{2^l l!}\left(\frac{d}{d\xi}\right)^l (\xi^2 - 1)^l.$$

Leaving the numerical coefficient aside, $P_4(\xi) \propto \left(\frac{d}{d\xi}\right)^4 (\xi^2 - 1)^4.$ The differentiation is not as hard as one could imagine, because $\left(\xi^2 - 1\right)^4 = \xi^8 - 4\xi^6 + 6\xi^4 - 4\xi^2 + 1,$ and the two lowest -power terms are not important, because they gradually disappear at sequential differentiation:

$$\left(\frac{d}{d\xi}\right)^1 (\xi^2 - 1)^4 = 8\xi^7 - 4\cdot 6\xi^5 + 6\cdot 4\xi^3 - 4\cdot 2\xi$$

$$\left(\frac{d}{d\xi}\right)^2 (\xi^2 - 1)^4 = 8\cdot 7\xi^6 - 4\cdot 6\cdot 5\xi^4 + 6\cdot 4\cdot 3\xi^2 - 4\cdot 2,$$

$$\left(\frac{d}{d\xi}\right)^3 (\xi^2 - 1)^4 = 8\cdot 7\cdot 6\xi^5 - 4\cdot 6\cdot 5\cdot 4\xi^3 + 6\cdot 4\cdot 3\cdot 2\xi,$$

$$\left(\frac{d}{d\xi}\right)^4 (\xi^2 - 1)^4 = 8\cdot 7\cdot 6\cdot 5\xi^4 - 4\cdot 6\cdot 5\cdot 4\cdot 3\xi^2 + 6\cdot 4\cdot 3\cdot 2$$

$$= 2^4 3\cdot(35\xi^4 - 30\xi^2 + 3),$$

so that, finally,

$$Y_4^0(\theta,\varphi) = const \times \left(35\cos^4\theta - 30\cos^2\theta + 3\right).$$

Problem F.6

Verify that the general uncertainty relation $\Delta A\cdot\Delta B \geq \frac{1}{2}\left|\left\langle [A,B]\right\rangle\right|$, when specified for spin component operators S_x and S_y, is satisfied for an electron with spin oriented along *z* axis (i.e. state $\left|\uparrow\right\rangle$).

Solution (Way 1):

For operators S_x and S_y we have $\left[S_x, S_y\right] = i\hbar S_z,$ so that for *any* state we must have

$$\Delta S_x \cdot \Delta S_y \geq \frac{\hbar}{2}\left|\left\langle S_z\right\rangle\right|.$$

Now, as has been discussed in class,

$$\left|\uparrow\right\rangle = \frac{1}{\sqrt{2}}\left(\left|\rightarrow\right\rangle - \left|\leftarrow\right\rangle\right) = \frac{1}{\sqrt{2}}\left(\left|\otimes\right\rangle + \left|\bullet\right\rangle\right),$$

so that for this state the spin orientation probabilities are as follows:

$$p_{\rightarrow} = p_{\leftarrow} = p_{\otimes} = p_{\bullet} = \frac{1}{2}, \quad p_{\uparrow} = 1, \quad p_{\downarrow} = 0.$$

From here,

$$\left\langle S_x \right\rangle = \left\langle S_y \right\rangle = 0, \quad \left\langle S_z \right\rangle = +\frac{\hbar}{2},$$

And, since $\left\langle S_x^2 \right\rangle = \left\langle S_y^2 \right\rangle = \left(\frac{\hbar}{2}\right)^2$,

$$\left\langle \left(S_x - \left\langle S_x \right\rangle\right)^2 \right\rangle = \left\langle S_x^2 \right\rangle - \left\langle S_x \right\rangle^2 = \left(\frac{\hbar}{2}\right)^2 - 0 = \frac{\hbar^2}{4},$$

$$\left\langle \left(S_y - \left\langle S_y \right\rangle\right)^2 \right\rangle = \left\langle S_y^2 \right\rangle - \left\langle S_y \right\rangle^2 = \left(\frac{\hbar}{2}\right)^2 - 0 = \frac{\hbar^2}{4}.$$

The r.m.s. deviations of components from the average are just square roots of their variances:

$$\Delta S_x = \frac{\hbar}{2}, \quad \Delta S_y = \frac{\hbar}{2},$$

so that $\Delta S_x \cdot \Delta S_y = \left(\frac{\hbar}{2}\right)^2$, $\frac{\hbar}{2}\left\langle S_z \right\rangle = \left(\frac{\hbar}{2}\right)^2$, so that the uncertainty relation is indeed satisfied.

Solution (Way 2):

A straightforward calculation of all averages using the matrix language:

$$\left\langle S_x \right\rangle = \left\langle \uparrow \right| S_x \left| \uparrow \right\rangle = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = 0,$$

$$\left\langle S_y \right\rangle = \left\langle \uparrow \right| S_y \left| \uparrow \right\rangle = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = 0,$$

$$\left\langle S_z \right\rangle = \left\langle \uparrow \right| S_z \left| \uparrow \right\rangle = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{\hbar}{2},$$

allows us to proceed directly to the calculation of component uncertainties and their product.

Though this method is more compact, it does not give such a revealing picture of our (spin-up) state as the first one.