PHY 308/578 **QUANTUM PHYSICS** **Spring 2006**

Homework #1 with Solutions

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Problem 1.1

A particle of mass m moves on a circular orbit of radius r about a central point to which it is attracted by force $F = \alpha r$ (α = const). Use classical mechanics augmented with the second Bohr's postulate ($pr = n\hbar$) to find the allowed values of:

(i) orbit radius,
(ii) particle velocity,
(iii) kinetic energy, and
(iv) potential energy.

Compare the last two results with each other and with the expression $\hbar\omega$, where ω is the frequency of 1D oscillations of the same system. Give a (very brief) discussion of your findings.

Solution:

Combining the classical equation of motion $mv^2/r = \alpha r$ with the Bohr's postulate, we get:

(i) $r_n = n^{1/2}x_0$, where $x_0 \equiv (\hbar/m\omega)^{1/2}$, and $\omega = (\alpha/m)^{1/2}$.
(ii) $v_n = \omega r_n = n^{1/2}\omega x_0$,
(iii) $T_n = mv_n^2/2 = n\,\hbar\omega/2$,
(iv) $U_n = \alpha r_n^2/2 = n\,\hbar\omega/2$.

The total energy of the system, $E_n = T_n + U_n = n\,\hbar\omega$. This means that the frequency of radiation at quantum transitions between two adjacent energy levels ($\Delta E = E_{n+1} - E_n$) is exactly equal to the classical frequency of this system (a "harmonic oscillator").

Problem 1.2

In class, we have calculated the Fourier amplitude distribution $a(k) = C\times\exp(-\frac{k^2}{2\sigma_k^2})$ corresponding to the Gaussian wave packet $\psi(x) = \exp(-\frac{x^2}{2\sigma_x^2})$, with $\sigma_x\sigma_k = 1$.

(i) Calculate the constant factor C in the expression for $a(k)$.

(ii) Find $a(k)$ for the wave packet $\psi(x,0) = \exp(ik_0x)\exp(-\frac{x^2}{2\sigma_x^2})$, where k_0 = const.

Solution:

(i) Direct integration, using the method discussed in class and the table integral $\int_{-\infty}^{+\infty}\exp(-\xi^2)d\xi = \sqrt{\pi}$, gives $C = \frac{\sigma_x}{\sqrt{2\pi}} = \frac{1}{\sqrt{2\pi}\sigma_k}$.

(ii) Since $a(k) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\psi(x,0)\exp(-kx)dx,$ we see that the multiplication of $\psi(x,0)$ by $\exp(ik_0x)$ just shifts the argument k by k_0, i.e. $a(k) = C\times\exp[-\frac{(k-k_0)^2}{2\sigma_k^2}]$.

Problem 1.3

Calculate the wave packet $\psi(x,0)$ corresponding to the distribution $a(k)=\frac{1}{(k-k_0)^2+\kappa^2}$, where k_0 and κ are constants.

Solution:

Using the table integral $\int_{-\infty}^{+\infty}\frac{\cos ax}{a^2+1}dx=\pi\exp(-|a|)$, we get $\psi(x,0)=\frac{\pi}{|\kappa|}\exp(ik_0x-|\kappa x|)$.

Problem 1.4

Find the phase and group velocities for waves with the following dispersion laws:

(i) $\lambda=\frac{c}{\sqrt{\nu^2-\nu_0^2}}$ (e.g., EM waves in plasma or inside a hollow waveguide),

(ii) $\nu=\sqrt{\frac{g}{2\pi\lambda}}$ (gravity waves on deep water).

c, ν_0 and g (and of course $\pi \cong 3.14$) are constants.

Solution:

In both cases, we should first rewrite the dispersion relation in the form $\omega = \omega(k)$ (using the relations $\nu=\omega/2\pi$, $\lambda=2\pi/k$), and then calculate $v_p = \omega/k$ and $v_g = \partial\omega/\partial k$. This procedure yields:

(i) $\omega=\sqrt{\omega_0^2+k^2c^2}$, $v_p = \frac{\sqrt{\omega_0^2+k^2c^2}}{k}$, $v_g = \frac{kc^2}{\sqrt{\omega_0^2+k^2c^2}}$, so that $v_p v_g = c^2$;

(ii) $\omega=\sqrt{gk}$, $v_p = \sqrt{\frac{g}{k}}$, $v_g = \frac{1}{2}\sqrt{\frac{g}{k}}$, so that $v_g = v_p/2$.