PHY 308/578 **QUANTUM PHYSICS** **Spring 2006**

Homework #3 with Solutions

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Problem 3.1 (10 points)
A 1D particle is in a ground state in an infinitely deep quantum well with $-a < x < +a$. Suddenly, the well walls are moved out to new positions $x = \pm b$ (with $b > a$). Find the probability of the particle being in the ground state of the new well. Analyze how this probability depends on the b/a ratio.

Solution: Since the Schrödinger equation says the wavefunction can only change with finite speed, its shape in the moment just after the quantum well expansion is the same as just before that instant:

$$u(x) = \begin{cases} \dfrac{1}{\sqrt{a}}\cos\dfrac{\pi x}{2a}, & |x| < a, \\ 0, & a < |x| < b. \end{cases}.$$

(Note that the well width is 2*a* rather than *a*, and, in contrast to the example made in class, is centered to zero.)

In order to solve the problem, we need to expand this *initial* wavefunction in series over the eigenfunctions $u_n(x)$ of the *finite* well. In particular, for the ground state ($n = 1$), $u_1(x) = \dfrac{1}{\sqrt{b}}\cos\dfrac{\pi x}{2b}$, and since we are using normalized functions $u(x)$ and $u_1(x)$, the expansion coefficient is just $A_1 = \int_{-b}^{+b} u(x)u_1(x)dx.$ The integration yields:

$$A_1 = \frac{1}{\sqrt{ab}}\int_{-a}^{+a}\cos\frac{\pi x}{2a}\cos\frac{\pi x}{2b}dx = \frac{1}{\sqrt{ab}}\int_{0}^{+a}\left(\cos(\frac{\pi x}{2a}+\frac{\pi x}{2b}) + \cos(\frac{\pi x}{2a}-\frac{\pi x}{2b})\right)dx = \frac{4}{\pi}\sqrt{\frac{a}{b}}\frac{1}{\left(1-\dfrac{a^2}{b^2}\right)}\cos\frac{\pi a}{2b}.$$

The required probability p_1 is just the square of A_1. The result shows that if $b \to a$, $p_1 \to 1$, because in this limit $1-\dfrac{a^2}{b^2} \approx 2\dfrac{b-a}{b}$, while $\cos\left(\dfrac{\pi a}{2b}\right) \approx \dfrac{\pi}{2}(b-a)$. On the other hand, if $b >> a$, then $p_1 \to (16/\pi^2)(a/b) << 1.$

Problem 3.2
Calculate the probability current ("flux") j for wave function

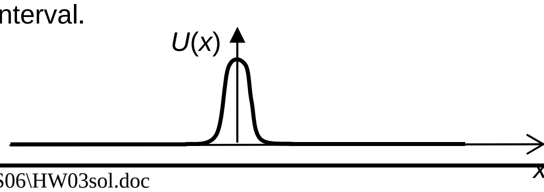
$$\psi(x,t) = [\exp(ikx) + R\cdot\exp(-ikx)]\exp(-i\omega t),$$

where R is generally a complex constant.

Solution: Plugging ψ into the probability current definition, $j = \dfrac{\hbar}{2im}\left(\psi^*\dfrac{\partial\psi}{\partial x} - \psi\dfrac{\partial\psi^*}{\partial x}\right),$

we get $j = j_0\left(1-|R|^2\right),$ where $j_0 \equiv \hbar k/m.$

Problem 3.3
Consider a "potential bump" localized on a finite segment of axis x (see Figure below), so that $U(x) = 0$ outside of this interval.



If the particle energy *E* is fixed, the most general solution of the Schrödinger equation outside of the bump range is

$$\psi(x,t) = \exp(-i\frac{E}{\hbar}t)\times\begin{cases} A\exp(ikx) + B\exp(-ikx), & \text{on the left of the bump,}\\ C\exp(ikx) + D\exp(-ikx), & \text{on the right of the bump,}\end{cases}$$

with $\hbar^2 k^2/2m = E.$ This is the typical elastic scattering problem which may be characterized by the following "scattering matrix":

$$S \equiv \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix},$$

whose components (which depend on the bump's size and shape) are the coefficients in the set of linear equations

$$C = S_{11}A + S_{12}D,$$
$$B = S_{21}A + S_{22}D.$$

Prove that for an arbitrary bump,

$$|S_{11}|^2 + |S_{21}|^2 = 1, \qquad |S_{12}|^2 + |S_{22}|^2 = 1.$$

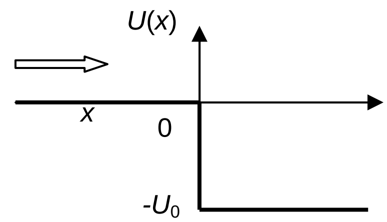
Solution: Calcutating probability currents (just as in Problem 3.1), we get:

$$j = \frac{\hbar k}{m}\begin{cases} |A|^2 - |B|^2, & \text{on the left of the bump,}\\ |C|^2 - |D|^2, & \text{on the right of the bump.}\end{cases}$$

Due to the probability conservation, these currents should be equal: $|A|^2 - |B|^2 = |C|^2 - |D|^2$. This equality should hold for any combination of incident wave amplitudes *A* and *D*. Plugging in the above expressions for *B* and *C*, and taking $A \neq 0$, $D = 0$, we get the first relation for the matrix elements, while taking $A = 0$, $D \neq 0$ yields the second one.

<u>Problem 3.4</u> (10 points)

Find the reflection coefficient $|R|^2$ for a quantum particle incident on a "negative" potential step (see Figure below), as a function of particle's energy *E*.



Solution: An examination of the solution carried out in class for posivive potential step shows that the solution is actually valid for any sign of U_0. Thus, $|R|^2 = \left(\frac{k-q}{k+q}\right)^2$, where now

$\frac{\hbar^2 q^2}{2m} = E + U_0$, so that $|R|^2 = \left(\frac{\sqrt{E} - \sqrt{E + U_0}}{\sqrt{E} + \sqrt{E + U_0}} \right)^2$, We see that the reflection coefficient $|R|^2 = 1$ only at E = 0, and rapidly decreases with the growth of particle energy.