

Problem 4.1

Find all components of the scattering matrix S defined in Homework Problem 3.3 (see also Gasiorowitz p. 90) for a potential barrier of height U_0 and thickness $2a$.

Solution:

The components of S are defined as the coefficients in the equations

$$C = S_{11}A + S_{12}D,$$

$$B = S_{21}A + S_{22}D,$$

where the wave amplitudes are defined by formulas:

$$u(x) = \begin{cases} A \exp(ikx) + B \exp(-ikx), & \text{on the left from the barrier,} \\ C \exp(ikx) + D \exp(-ikx), & \text{on the right from the barrier.} \end{cases}$$

These components do not depend on the incident wave amplitudes A and D , so we can use any combination of those. In order to find S_{11} and S_{21} , it is sufficient to take $D = 0$ and $A = 1$, i.e. consider the problem we have solved in class (with the notation change: $T \rightarrow C$, $R \rightarrow B$). Solving the system of 4 linear equations (which express 4 boundary conditions discussed in class) we get

$$S_{11} \equiv C \Big|_{D=0, A=1} = T = \exp(-2ika) \frac{2k\kappa}{2k\kappa \cosh 2\kappa a - i(k^2 - \kappa^2) \sinh 2\kappa a},$$

$$S_{21} \equiv B \Big|_{D=0, A=1} = R = i \exp(-2ika) \frac{(k^2 + \kappa^2) \sinh 2\kappa a}{2k\kappa \cosh 2\kappa a - i(k^2 - \kappa^2) \sinh 2\kappa a}.$$

Vice versa, in order to find S_{12} and S_{22} , we may take $A = 0$, $D = 1$, and repeat all the calculations. However, this effort may be saved by noticing that the problem is symmetric with respect to the simultaneous replacements $k \leftrightarrow -k$, $\kappa \leftrightarrow -\kappa$, $A \leftrightarrow D$, $B \leftrightarrow C$. As a result, we can immediately write:

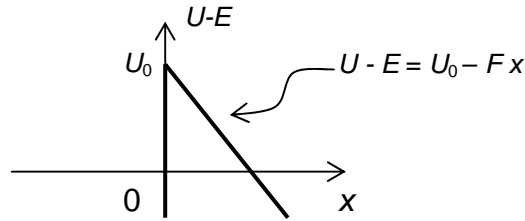
$$S_{12} \equiv C \Big|_{D=1, A=0} = i \exp(2ika) \frac{(k^2 + \kappa^2) \sinh 2\kappa a}{2k\kappa \cosh 2\kappa a + i(k^2 - \kappa^2) \sinh 2\kappa a},$$

$$S_{22} \equiv B \Big|_{D=1, A=0} = \exp(2ika) \frac{2k\kappa}{2k\kappa \cosh 2\kappa a + i(k^2 - \kappa^2) \sinh 2\kappa a}.$$

Note that $S_{11}S_{12}^* + S_{21}S_{22}^* = 0$. One can prove that this is one more general property of the scattering matrix.

Problem 4.2

Use the WKB approximation to find the transparency of the triangular (“Fowler-Nordheim”) barrier sketched below. Analyze the dependence of the transparency on the force F which creates the potential slope.



Solution: As discussed in class, in the WKB approximation, the tunnel barrier transparency

$$|T|^2(E) = \exp\left(-\frac{2}{\hbar} \int \sqrt{2m[U(x) - E]} dx,\right)$$

where the integral should be limited to the “classically forbidden” region $U(x) > E$. In our case,

$$\begin{aligned} |T|^2(E) &= \exp\left(-\frac{2}{\hbar} \int_0^{a=U_0/F} \sqrt{2m[U_0 - Fx]} dx\right) = \exp\left(-\frac{2}{\hbar} \sqrt{2mU_0} \int_0^a \sqrt{1 - \frac{x}{a}} dx\right) \\ &= \exp\left(-\frac{2}{\hbar} \sqrt{2mU_0} a \int_0^1 \sqrt{1 - \xi} d\xi\right) = \exp\left(-\frac{4}{3\hbar} \sqrt{2mU_0} a\right) = \exp\left(-\frac{4\sqrt{2m}^{1/2} U_0^{3/2}}{3\hbar F}\right). \end{aligned}$$

The result (first derived by R. Fowler and F. Nordheim in 1928, i.e. very soon after the Schödinger equation had been suggested) shows that the dependence of transparency logarithm of the reciprocal force ($1/F$) should follow a straight line. This is in a very good agreement with experiments for, e.g., field emission of electrons from metals. (In this case, U_0 is the “work function”, for most metals between 4 and 5 eV, while the force is the electron charge times the applied electric field.)