Homework #7 with Solutions

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Problem 7.1

For the *n*-th eigenstate of a 1D harmonic oscillator, find averages $\langle x\rangle, \langle x^2\rangle, \langle p\rangle,$ and $\langle p^2\rangle$ and the product $\Delta x\Delta p$ of the r.m.s. fluctuations of the coordinate and momentum.

Hint 1: Instead of using the explicit form of eigenfunctions $u_n(x)$, with their Hermite polynomials, express operators x and p as linear combinations of the creation and annihilation operators a^+, a, and use their properties discussed in class, in particular the results for a^+u_n and au_n.

Hint 2. Remember that the eigenfunctions u_n are "orthonormal": $\int_{-\infty}^{+\infty} u_n^*(x)u_{n'}(x)dx = \delta_{n,n'}$.

Solution:

Solving the equations, which define a^+, a, for x and p, we get:

$$x = \sqrt{\frac{\hbar}{2m\omega}}(a + a^+), \quad p = \frac{1}{i}\sqrt{\frac{\hbar m\omega}{2}}(a - a^+).$$

Plugging the first of these formulas into the definition of the average $\langle x\rangle = \int_{-\infty}^{+\infty} u_n^*(x)xu_n(x)dx,$ and using the relations discussed in class, $au_n = \sqrt{n}u_{n-1},\ a^+u_n = \sqrt{n+1}u_{n+1},$ we get:

$$\langle x\rangle = \sqrt{\frac{\hbar}{2m\omega}}\int_{-\infty}^{+\infty} u_n^*(x)(a + a^+)u_n(x)dx = \sqrt{\frac{\hbar}{2m\omega}}\int_{-\infty}^{+\infty} u_n^*(x)\left[\sqrt{n}u_{n-1}(x) + \sqrt{n+1}u_{n+1}(x)\right]dx$$

$$= \sqrt{\frac{\hbar}{2m\omega}}\left[\sqrt{n}\int_{-\infty}^{+\infty} u_n^*(x)u_{n-1}(x)dx + \sqrt{n+1}\int_{-\infty}^{+\infty} u_n^*(x)u_{n+1}(x)dx\right].$$

Due to the orthogonality of the eigenfunctions with different indices, both integrals equal zero and hence $\langle x\rangle = 0.$ This could be anticipated because of the symmetry of the problem.

However, the average of x^2 is not equal zero:

$$\langle x^2\rangle = \frac{\hbar}{2m\omega}\int_{-\infty}^{+\infty} u_n^*(x)(a + a^+)^2 u_n(x)dx = \frac{\hbar}{2m\omega}\int_{-\infty}^{+\infty} u_n^*(x)\left(a^2 + a^{+2} + aa^+ + a^+a\right)u_n(x)dx.$$

The sequential application of operators yields:

$$\langle x^2\rangle = \frac{\hbar}{2m\omega}\sqrt{n(n-1)}\int_{-\infty}^{+\infty} u_n^*(x)u_{n-2}(x)dx + \sqrt{(n+1)(n+2)}\int_{-\infty}^{+\infty} u_n^*(x)u_{n+2}(x)dx$$

$$+ (n+1)\int_{-\infty}^{+\infty} u_n^*(x)u_n(x)dx + n\int_{-\infty}^{+\infty} u_n^*(x)u_n(x)dx] = (2n+1)\frac{\hbar}{2m\omega}.$$

For the ground state (n = 0), this result reduces to the fundamental formula $\langle x^2\rangle = \frac{\hbar}{2m\omega}$ which we obtained in class directly from the Gaussian wavefunction of that state. In the excited states (n > 0), $\langle x^2\rangle$ grows linearly with n. This result could be expected, because the average is directly related to the average potential energy of the oscillator: $\langle U\rangle = \frac{m\omega^2}{2}\langle x^2\rangle = \frac{\hbar\omega}{4}(2n+1).$ This is just a half of $E_n = \hbar\omega(n+1/2),$ as could be expected from classical mechanics.

The absolutely similar calculations for momentum give $\langle p\rangle = 0, \langle p^2\rangle = \frac{\hbar m\omega}{2}(2n+1),$ so that the average kinetic energy of the oscillator, $\langle K\rangle = \frac{\langle p^2\rangle}{2m} = \frac{\hbar\omega}{4}(2n+1) = \langle U\rangle = \frac{E_n}{2}.$

The Heisenberg uncertainty product $\Delta x\Delta p = \left(\langle x^2\rangle\langle p^2\rangle\right)^{1/2} = (2n+1)\frac{\hbar}{2}.$ Quite naturally, the uncertainty is smallest in the ground state.

<u>Problem 7.2</u>

A harmonic oscillator is initially prepared in state $\psi(x,0) = \frac{1}{\sqrt{2}}\left[u_0(x) + u_1(x)\right].$ Find the time evolution of the (ensemble) average $\langle x\rangle$.

Solution: Time evolution of each eigenfunction follows $\exp(-iE_n t/\hbar),$ with $E_n = \hbar\omega(n+1/2)$ so that the whole wavefunction evolves as

$$\psi(x,t) = \frac{1}{\sqrt{2}}\left[u_0(x)\exp(-i\frac{E_0}{\hbar}t) + u_1(x)\exp(-i\frac{E_1}{\hbar}t)\right] = \frac{1}{\sqrt{2}}\exp(-i\frac{\omega}{2}t)\left[u_0(x) + u_1(x)\exp(-i\omega t)\right].$$

Now, we can calculate the requested average:

$$\langle x\rangle = \int_{-\infty}^{+\infty}\psi^*(x,t)x\psi(x,t)dx = \frac{1}{2}\int_{-\infty}^{+\infty}\left[u_0^*(x) + u_1^*(x)\exp(+i\omega t)\right]x\left[u_0(x) + u_1(x)\exp(-i\omega t)\right]dx$$

$$= \frac{1}{2}\left[\int_{-\infty}^{+\infty}u_0^*(x)xu_0(x)dx + \int_{-\infty}^{+\infty}u_1^*(x)xu_1(x)dx + \exp(-i\omega t)\int_{-\infty}^{+\infty}u_0^*(x)xu_1(x)dx + \exp(i\omega t)\int_{-\infty}^{+\infty}u_1^*(x)xu_0(x)dx\right].$$

As we already know from Problem 7.1, the first two integrals equal zero. However, the last ones are not:

$$\int_{-\infty}^{+\infty}u_1^*(x)xu_0(x)dx = \sqrt{\frac{\hbar}{2m\omega}}\int_{-\infty}^{+\infty}u_1^*(x)\left(a + a^+\right)u_0(x)dx = \sqrt{\frac{\hbar}{2m\omega}}\int_{-\infty}^{+\infty}u_1^*(x)u_1(x)dx = \sqrt{\frac{\hbar}{2m\omega}},$$

$$\int_{-\infty}^{+\infty}u_0^*(x)xu_1(x)dx = \sqrt{\frac{\hbar}{2m\omega}}\int_{-\infty}^{+\infty}u_0^*(x)\left(a + a^+\right)u_1(x)dx = \sqrt{\frac{\hbar}{2m\omega}}\int_{-\infty}^{+\infty}u_0^*(x)u_0(x)dx = \sqrt{\frac{\hbar}{2m\omega}}.$$

As a result,

$$\langle x\rangle = \frac{1}{2}\sqrt{\frac{\hbar}{2m\omega}}\left[\exp(+i\omega t) + \exp(-i\omega t)\right] = \sqrt{\frac{\hbar}{2m\omega}}\cos\omega t.$$

It is a remarkable property of quantum mechanics that a superposition of two states, each of which separately would give $\langle x\rangle = 0,$ yields $\langle x\rangle \neq 0.$ Essentially, this is the time equivalent of the wave interference in space. We have discussed in class another example of this phenomenon: the coherent (Glauber) state.