PHY 501 **Classical Mechanics** Fall 2006

Homework 04 with Solution

Problem 4.1.

(a) Include nonlinear term αx^3 into the equation describing the parametric excitation problem discussed in class; find the corresponding addition to the reduced (van der Pol) equations.

Solution: Looking for oscillations in the usual form $x = a\cos\psi$, $\psi \equiv \omega t + \phi$, we get the following additional contributions to the right-hand parts of the reduced equations:

$$-\frac{1}{2\pi\omega}\int_0^{2\pi}[-\alpha(a\cos\psi)^3]\sin\psi\, d\psi = 0,$$

$$-\frac{1}{2\pi\omega}\int_0^{2\pi}[-\alpha(a\cos\psi)^3]\cos\psi\, d\psi = -\frac{3}{8}\frac{\alpha a^3}{\omega},$$

so that the equations take the form

$$\dot{a} = -\delta a + \frac{m\omega}{4} a\sin 2\varphi, \qquad \xi(a) \equiv \xi - \frac{3}{8}\frac{\alpha a^2}{\omega}. \qquad (1)$$
$$a\dot{\varphi} = -\xi(a)a + \frac{m\omega}{4}\cos 2\varphi,$$

(b) Find the stationary amplitude a_0 of parametric oscillations; sketch and discuss the $a_0(\xi)$ dependence.

Solution: In any fixed point, $\dot{a} = \dot{\phi} = 0.$ Solving Eqs. (*) for this case, we get three values:

$$a_1 = 0, \qquad a_{2,3}^2 = \frac{8\omega}{3\alpha}[\xi \pm \xi_t], \qquad \xi_t \equiv \sqrt{\left(\frac{m\omega}{4}\right)^2 - \delta^2}. \qquad (2)$$

The first of them describes the system at rest (no parametric excitation), the dependences $a_{2,3}(\xi)$ are sketched in Fig. 1 for the case $\alpha > 0$, $m\omega/4 > \delta$. (If $\alpha < 0$, the situation is similar, besides that the curves $a_{2,3}(\xi)$ are bent toward negative rather than positive ξ. If $m\omega/4 < \delta$, there is no parametric excitation at any ξ.)

The results show that the Van der Pol approximation cannot give the absolute maximum for the parametric excitation amplitude. (Going after it, we would need to take $\xi \sim \omega$, where this approximation is not valid. Also note that for the nontrivial fixed points $a_{2,3}$, the following formulas are valid:

$$\xi(a_{2,3}) = \mp\xi_t, \ \sin 2\varphi_{2,3} = \delta/(m\omega/4), \cos 2\varphi_{2,3} = \mp\xi_t/(m\omega/4). \qquad (3)$$

These expressions simplify the fixed point stability analysis (see below).

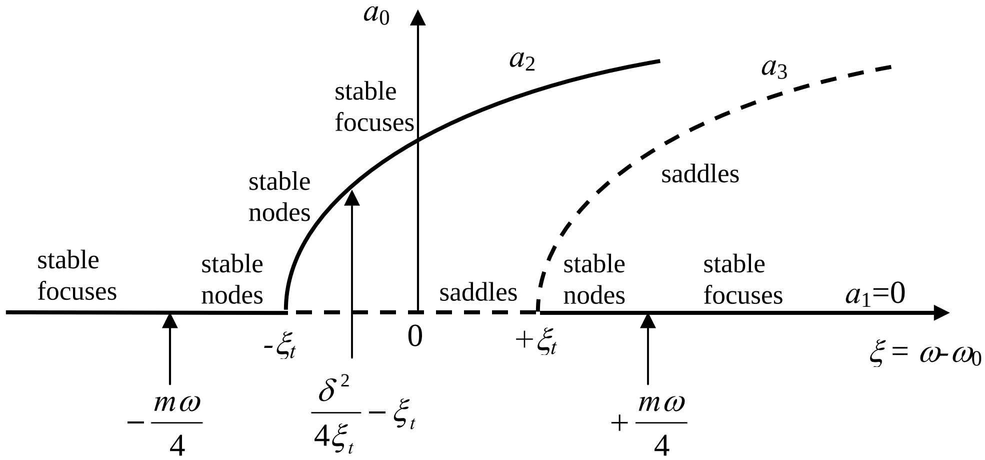

Fig. 1. Fixed points of the reduced equations (schematically) and their type.

(c) Find the type and stability of each fixed point of the reduced equations.

Solution:

For the *trivial* fixed point a_1 = 0, this problem has been solved in class using $\{u, v\}$ variables . If the modulation depth m is large enough,

$$\frac{m\omega}{4} > \delta,$$

there is a finite range of frequencies, $-\xi_t < \xi < +\xi_t$, where the largest root λ_+ is positive (i.e., the trivial point is unstable and parametric oscillations are excited). In this case, the fixed point is a <u>saddle</u>. On the contrary, at

$$\xi_t < |\xi| < \frac{m\omega}{4},$$

(i.e. outside of, but adjacent to the excitation range) both roots are real and negative, so we are dealing with a <u>stable node</u>. Finally, at even larger detuning, $|\xi| > \xi_t$, both roots λ have a negative real part (Re$\lambda_\pm$ = -δ) and finite imaginary parts, so that the trivial fixed point is a <u>stable focus</u>. The stability is indicated in Fig. 1 (drawn for the case $\alpha > 0$).

Near the *non-trivial* fixed points ($a_{2,3} \neq 0$), the reduced equations are regular in both forms, $\{a, \varphi\}$ and $\{u, v\}$, and we can use either of them for the stability analysis. Calculations are a little bit simpler in the $\{a, \varphi\}$ format. Taking $a = a_0 + \tilde{a}, \varphi = \varphi_0 + \tilde{\varphi}$, and linearizing the equations with respect to small $\tilde{a}, \tilde{\varphi}$, and using the we get:

$$\dot{\tilde{a}} = 2\xi(a_{2,3})a_{2,3}\tilde{\varphi},$$
$$a_{2,3}\dot{\tilde{\varphi}} = -\xi'(a_{2,3})a_{2,3}\tilde{a} - 2\delta a_{2,3}\tilde{\varphi}.$$

Solving the characteristic equation for this linear system, we get

$$\lambda_\pm = -\delta \pm \sqrt{\delta^2 - 2\xi(a_{2,3})\xi'(a_{2,3})a_{2.3}}\ . \qquad (4)$$

Let us assume that $\alpha > 0$. (For the opposite case, the final result topology is similar.) In this case $\xi'(a_{2,3})a_{2,3} = -(3/4)\alpha a_{2,3}^2/\omega < 0$. For the lower branch of fixed points (dashed curve in Fig. 1), $\xi(a_3) = +\xi_t$, so that the last term under the square root in Eq. (4) is positive. Hence the both roots $\lambda_\pm$ are real, and one of them is positive while another one is negative. Thus these fixed points are (unstable) saddles.

For the upper branch (solid curve in Fig. 1), $\xi(a_2) = -\xi_t$. Because of this, the last term under the square root is always negative, so that $\mathrm{Re}\,\lambda_\pm < 0$ and these fixed points are always stable. The fixed point type here depends on the detuning: if $2\xi(a_2)\xi'(a_2)a_2 > \delta^2$, the expression under the square root is negative and the points are stable focuses. Since $\xi'(a_2)a_2 = 2[\xi(a) - \xi]$, and $\xi(a_2) = -\xi_t$, the condition for that may be re-written as

$$\xi > \frac{\delta^2}{4\xi_t} - \xi_t$$

In the opposite case they are stable nodes.

(d) Sketch the Poincare phase plane.

Solution:

- The case $\xi < -\xi_t$ is simple - no parametric excitation, just a stable focus or node in the origin – see Fig. 2a below.

- Within the range $-\xi_t < \xi < +\xi_t$, we should merge a saddle near $a = 0$ (it was discussed in class) with two stable focuses (or nodes) near two non-trivial stable fixed points $\{a_2, \varphi_2\}$ and $\{a_2, \varphi_2+\pi\}$. As a result, we get something like Fig. 2b.

- If $+\xi_t < \xi$, we should accommodate five fixed points: a stable focus or node at the origin, two unstable saddles corresponding to $a_3 \neq 0$, and two stable focuses corresponding to $a_2 > a_3$. The result is shown in Fig. 2c.

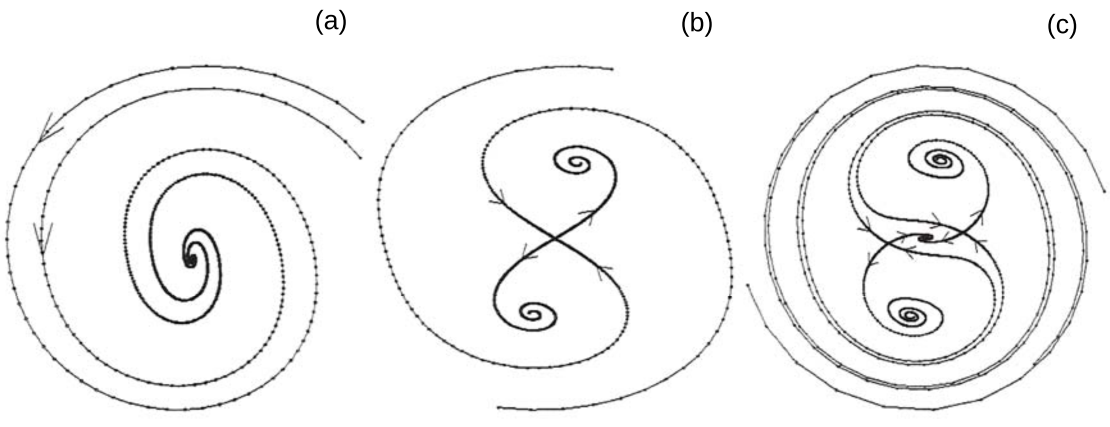

Fig. 2. Poincare plane for a parametric oscillator with nonlinear detuning.