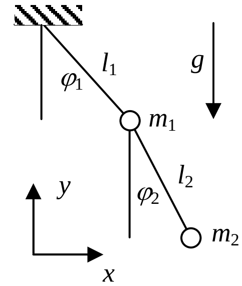
PHY 501 **Classical Mechanics** Fall 2006

Homework 05 with Solutions

Problem 5.1. For a double pendulum confined to a vertical plane, with parameters shown in Fig. below, find possible frequencies of small sinusoidal oscillations, and the corresponding distribution coefficients. Sketch both oscillation modes.

Solution:



In order to retain means of sanity checking (see below), it is always better to do calculations for arbitrary parameters, until this starts to lead to bulk expressions. With such generalization, and using angles φ_1 and φ_2 (see Figure on the right) as the generalized coordinates, we have:

$$\begin{aligned}
x_1 &= l_1 \sin\varphi_1, \quad y_1 = -l_1 \cos\varphi_1, \\
x_2 &= l_1 \sin\varphi_1 + l_2 \sin\varphi_2, \quad y_2 = -l_1 \cos\varphi_1 - l_2 \cos\varphi_2, \\
\dot{x}_1 &= l_1\dot{\varphi}_1 \cos\varphi_1, \quad \dot{y}_1 = l_1\dot{\varphi}_1 \sin\varphi_1, \\
\dot{x}_2 &= l_1\dot{\varphi}_1 \cos\varphi_1 + l_2\dot{\varphi}_2 \cos\varphi_2, \quad y_2 = l_1\dot{\varphi}_1 \sin\varphi_1 + l_2\dot{\varphi}_2 \sin\varphi_2,
\end{aligned}$$

so that the Lagrangian is

$$L = T - U = \left[\frac{m_1}{2}(x_1^2 + y_1^2) + \frac{m_2}{2}(x_2^2 + y_2^2)\right] - \left[m_1 g y_1 + m_2 g y_2\right]$$

$$= \left[\frac{m_1 + m_2}{2} l_1^2\dot{\varphi}_1^2 + \frac{m_2}{2} l_2^2\dot{\varphi}_2^2 + m_2 l_1 l_2 \dot{\varphi}_1\dot{\varphi}_2 \cos(\varphi_1 - \varphi_2)\right] + \left[(m_1 + m_2) g l_1 \cos\varphi_1 + m_2 g l \cos\varphi_2\right].$$

Expanding the Lagrangian into the Taylor series in small angles $\varphi_{1,2}$ and their time derivatives, and keeping only the lowest (second) order terms, we get

$$L = \frac{m_1 + m_2}{2} l_1^2\dot{\varphi}_1^2 + \frac{m_2}{2} l_2^2\dot{\varphi}_2^2 + m_1 m_2 l_1 l_2 \dot{\varphi}_1\dot{\varphi}_2 - \frac{m_1 + m_2}{2} l_1 g\varphi_1^2 - \frac{m_2}{2} l_2 g\varphi_2^2 + const,$$

giving the following linear equations of motion:

$$\begin{aligned}
&(m_1 + m_2) l_1^2 (\ddot{\varphi}_1 + \Omega_1^2\varphi_1) + m_2 l_1 l_2 \ddot{\varphi}_2 = 0, \\
&m_2 l_1 l_2 \ddot{\varphi}_1 + m_2 l_2^2 (\ddot{\varphi}_2 + \Omega_2^2\varphi_2) = 0,
\end{aligned}$$

where $\Omega_{1,2}^2 \equiv g / l_{1,2}$. (Alternatively, we could get these the same result from the general Lagrangian equations of motion by their linearization with respect to small $\varphi_{1,2}$.)

Looking for the solution in the usual form $\varphi_{1,2} = C_{1,2} e^{\lambda t}$, we get the following set of two linear, homogeneous equations for the distribution coefficients $C_{1,2}^{\pm}$ and characteristic exponents λ:

$$\begin{aligned}
&C_1 (m_1 + m_2) l_1^2 (\lambda^2 + \Omega_1^2) + C_2 m_2 l_1 l_2 \lambda^2 = 0, \\
&C_1 m_2 l_1 l_2 \lambda^2 + C_2 m_2 l_2^2 (\lambda^2 + \Omega_2^2) = 0.
\end{aligned} \qquad (1)$$

In order for these equations to be self-consistent, the following characteristic equation

$$(\lambda^2 + \Omega_1^2)(\lambda^2 + \Omega_2^2) - \frac{m_2}{m_1 + m_2}\lambda^4 = 0 \qquad (2)$$

should be satisfied. Since the calculation of λ from this equation in the general case leads to a somewhat bulky formula, this is a good point to stop and check our calculations for obvious particular cases.

(i) If $m_2 / m_1 \to 0, l_1 \sim l_2$, then the solution of Eq. (2) is obvious: $\lambda_1 = \pm\Omega_1, \lambda_2 = \pm\Omega_2$, i.e. the pendula are independent. This is natural, because the heavy upper pendulum, because of its large mass, does not "feel" oscillations of the lower pendulum. At the same time, slow oscillations of the upper pendulum do not affect fast oscillations of its lower counterpart.

(ii) On the other hand, if $m_2 / m_1 \to \infty, l_1 \sim l_2$, then the result is quite different. For the lowest-frequency ("soft") oscillation mode, we can replace the factor $m_2 /(m_1 + m_2)$ in Eq. (2) with unity and readily get $\lambda_-^2 = g/(l_1 + l_2)$. This is also very natural, because in this limit the main mode of oscillations is the aligned motion of both pendula, with the total length (l_1 + l_2). (A good optional exercise: find and interpret the hard mode of oscillations in this limit.)

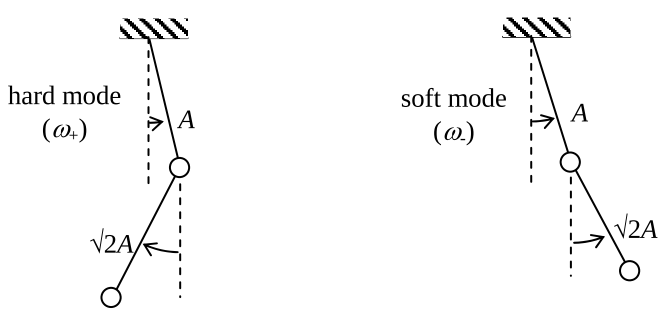
Now when we are sure our results are sensible, let us move to our particular case ($l_1 = l_2 = l$, $m_1 = m_2 = m$, i.e. $\Omega_1 = \Omega_2 \equiv \Omega$). In this case, Eq. (2) can be readily solved to give the following values of the "hard" (+) and "soft" frequencies of the system:

$$\omega_\pm^2 \equiv -\lambda_\pm^2 = \Omega^2\left(2 \pm \sqrt{2}\right).$$

Plugging these values of $\lambda_\pm^2$, one by one, into any of Eqs. (1), for the distribution coefficients (or rather their ratios) we get:

$$\frac{C_2^\pm}{C_1^\pm} = \mp\sqrt{2} \approx \mp 1.4,$$

so that the oscillation modes look as sketched (crudely) in Fig. below:



Problem 5.2. We know that the second term in the planetary problem's Lagrangian

$$L = \frac{m}{2}\dot{r}^2 + \frac{m}{2}r^2\dot{\varphi}^2 - U(r)$$

equals $L_z^2 / 2mr^2$. Explain why it cannot be, after this substitution, merged with $U(r)$ in the above expression. (This would of course give the effective 1D potential energy $U(r) - L_z^2 / 2mr^2$, substantially different from the (correct) $U_{ef}(r) = U(r) + L_z^2 / 2mr^2$ calculated in class.) Indeed, we have carried out an apparently similar transformation of the Lagrangian for the bead-on-rotating-ring problem (see in the very end of Sec. 3.1 of the lecture notes); why cannot the same trick work for the planetary problem?

Solution:

Deriving the Lagrangian equations in Chapter 2, we have made a commitment to treat generalized velocities (in the planetary case, $\dot{r}$ and $\dot{\varphi}$) as variables independent of the generalized coordinates (r and φ). Expressing $(m/2)r^2\dot{\varphi}^2$ as a function of r ($L_z^2 / 2mr^2$) in the Lagrangian function *before* its differentiation, we would violate this commitment, and as a result get wrong Lagrangian equations of motion. On the contrary, *after* these equations have been (correctly) obtained, we may make whichever substitutions we like, like we did in class. (See Eqs. (4.31)-(4.32) of the lecture notes.)

Concerning the bead-on-a-ring problem considered in Section 3.1, there we have indeed merged the kinetic-energy term $(m/2)R^2\omega^2 \sin^2\theta$ with the potential energy right in *L*, i.e. *before* obtaining the Lagrangian equation of motion. However, this term does not contain any generalized velocities, and hence that operation was legitimate.