PHY 501

Classical Mechanics

Fall 2006

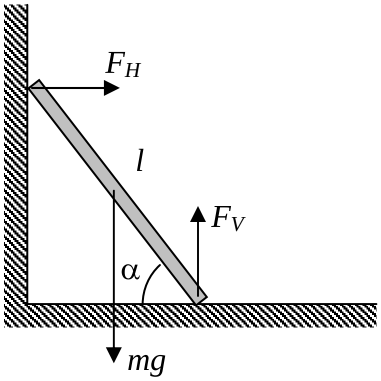
Homework 08 with Solution

Problem 8.1. For the "sliding ladder" problem started in class (see Sec. 5.9 of the lecture notes), assuming that the ladder starts sliding from a virtually vertical position with a negligible initial velocity:

(a) find the critical value α_c of angle α at which the ladder loses contact with the vertical wall;

(b) write the equations of motion and all available conservation laws for the further motion (at $\alpha < \alpha_c$);

(c) would the right end of ladder lose the contact with the floor before its left end hits the ground?



Solution:

(a) The ladder separates from the wall when the (horizontal) force F_H applied to the ladder from the wall (see Fig. above) turns to zero. The force may be found from the horizontal component of the 2nd Newton law for the center of mass:

$$F_H = M\ddot{X} = -\frac{Ml}{2}(\ddot{\alpha}\sin\alpha + \dot{\alpha}^2\cos\alpha).$$

We may express $\dot{\alpha}^2$ via α from using the first integral of motion (see Eq. (68) of the lecture notes) which, for our particular initial conditions ($E = Mgl/2$), takes the form

$$\dot{\alpha}^2 = 2\omega^2(1-\sin\alpha), \quad \omega^2 \equiv 3g/2l. \qquad (1)$$

Also, differentiating this equation over time gives us an expression for $\ddot{\alpha}$ as a function of α:

$$\ddot{\alpha} = -\omega^2\cos\alpha$$

(which is essentially the equation of motion of the system and could be obtained in the usual way from its Lagrangian). Plugging these expressions into the condition F_H=0, we get the following equation for the critical angle α_c:

$$\omega^2\left(-\cos\alpha_c\sin\alpha_c + 2(1-\sin\alpha_c)\cos\alpha_c\right) = 0,$$

which may be readily solved:

$$\alpha_c = \arcsin\frac{2}{3} \approx 42^\circ. \qquad (2)$$

For what follows, we will also need the value of $\dot\alpha^2$ at this angle, which may be obtained from Eqs. (1) and (2):

$$\dot\alpha^2\Big|_{\alpha=\alpha_c} = \frac{3g}{l}(1-\sin\alpha_c) = \frac{g}{l}. \qquad (3)$$

(b) At $\alpha < \alpha_c$, the constraint $X = (l/2)\cos\alpha$ is no longer valid, but the condition $Y = (l/2)\sin\alpha$ is still intact (if the right end of the ladder stays in contact with the floor). The latter condition may be used to exclude Y and $\dot Y$ from the Lagrangian, but $\dot X$ remains there:

$$L = \frac{M}{2}\left[\dot X^2 + (\frac{l}{2}\dot\alpha\cos\alpha)^2\right] + \frac{Ml^2}{24}\dot\alpha^2 - Mg\frac{l}{2}\sin\alpha, \qquad (4)$$

so that the system should be considered now as having two degrees of freedom, X and α.

The equation of motion along the former coordinate is now very simple: $\ddot X = 0$. (This is very natural, because there is no more horizontal force applied to the ladder.) The corresponding first integral of motion,

$$\dot X = const \qquad (5)$$

is just the law of conservation of the horizontal linear momentum.

For the second generalized coordinate, angle α, Eq. (4) yields the following Lagrangian equation of motion:

$$\frac{d}{dt}[\dot\alpha\cos^2\alpha + \frac{1}{3}\dot\alpha] + \dot\alpha^2\sin\alpha\cos\alpha + \frac{2g}{l}\cos\alpha = 0,$$

i.e.

$$\ddot\alpha(\cos^2\alpha + \frac{1}{3}) - \dot\alpha^2\sin\alpha\cos\alpha + \frac{2g}{l}\cos\alpha = 0. \qquad (6)$$

The first integral of this equation,

$$\frac{1}{2}\dot\alpha^2(\cos^2\alpha + \frac{1}{3}) + \frac{2g}{l}\sin\alpha = const. \qquad (7)$$

is essentially a combination of the laws of conservation of energy

$$E = \frac{M}{2}\dot X^2 + \frac{Ml^2}{8}(\dot\alpha\cos\alpha)^2 + \frac{Ml^2}{24}\dot\alpha^2 + Mg\frac{l}{2}\sin\alpha,$$

and momentum (5). The constant in Eq. (7) may be found from Eqs. (2) and (3) that play the role of initial conditions for this stage of the process:

$$\frac{1}{2}\dot\alpha^2(\cos^2\alpha + \frac{1}{3}) + \frac{2g}{l}\sin\alpha = \left[\frac{1}{2}\dot\alpha^2(\cos^2\alpha + \frac{1}{3}) + \frac{2g}{l}\sin\alpha\right]_{\alpha=\alpha_c} = \frac{16}{9}\frac{g}{l}. \qquad (8)$$

(c) The (vertical) force applied to the ladder from the floor may be calculated from the vertical component of the 2nd Newton law, $M\ddot{Y} = F_V - Mg$ (see Fig. above), giving

$$F_V = Mg + \frac{Ml}{2}\frac{d^2}{dt^2}\sin\alpha = Mg + \frac{Ml}{2}(\ddot{\alpha}\cos\alpha - \dot{\alpha}^2\sin\alpha).$$

Plugging in $\ddot{\alpha}$ from Eq. (6) and then $\dot{\alpha}^2$ from Eq. (8) into this expression, we get the force as a function of angle α:

$$F_V = Mg\, f(\alpha), \quad f(\alpha) = 1 + \frac{3}{(1+3\cos^2\alpha)}\left[2\sin\alpha\left(\frac{8}{9} - \sin\alpha\right)\left(\frac{3\cos\alpha}{1+3\cos^2\alpha} - 1\right) - \cos^2\alpha\right].$$

This function is positive for all angles within the range of the second stage of motion $0 < \alpha < \alpha_c$ (see the MathCAD plot below) and hence the right edge of the ladder does retain its contact with the floor all the way until the ladder drops flat.