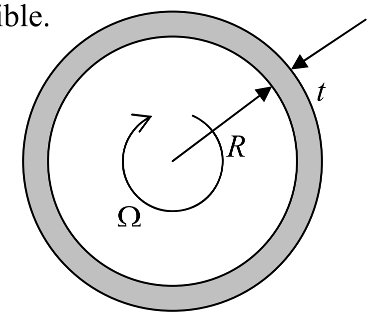
PHY 501 **Classical Mechanics** Fall 2006

Homework #9 Solutions

Problem 9.1. Calculate the extension ΔR of a thin, long, round cylindrical pipe rotated with constant angular velocity Ω about its axis (see Figure below) in terms of the elastic moduli E and σ. Pressure both inside and outside the pipe is negligible.



Solution:

In this case, we can write the static equilibrium equation (6.36) as

$$\vec{\nabla}\cdot(\vec{\nabla}\cdot\vec{s})-\frac{1-2\sigma}{2(1-\sigma)}\vec{\nabla}\times(\vec{\nabla}\times\vec{s})=-C\vec{r},\quad \text{where } C\equiv\rho\Omega^2\frac{(1+\sigma)(1-2\sigma)}{E(1-\sigma)}. \qquad (1)$$

in the non-inertial reference frame rotating with the pipe, were we have neglected the bulk gravity force, but added the inertial centrifugal "force". Because of the evident symmetry of the problem, $\vec{s}=\vec{n}_r s(r)$, so that the second term in the LHP of Eq. (1) vanishes, while the remaining terms have only one (radial) component:

$$\left[\frac{1}{r}(rs)'\right]'=-Cr, \qquad (2)$$

where the prime means differentiation over r. Integrating this equation twice, we get

$$s(r)=-\frac{1}{8}Cr^3+ar+\frac{b}{r}, \qquad (3)$$

where a and b are integration constants. From Eq. (3), using Eq. (6.8) of the lecture notes, we can readily calculate diagonal components of the strain tensor:

$$u_{rr}=\frac{\partial s}{\partial r}=-\frac{3}{8}Cr^2+a-\frac{b}{r^2},\quad u_{\phi\phi}=\frac{s}{r}=-\frac{1}{8}Cr^2+a+\frac{b}{r^2},\quad u_{zz}=0.$$

Applying the Hooke's law, we get the radial component of the stress tensor:

$$\sigma_{rr}=\frac{E}{1+\sigma}\left(u_{rr}+\frac{\sigma}{1-2\sigma}\sum_l u_{ll}\right)=\frac{E}{1+\sigma}\left[\left(-\frac{3}{8}Cr^2+a-\frac{b}{r^2}\right)+\frac{\sigma}{1-2\sigma}\left(-\frac{1}{2}Cr^2+2a\right)\right].$$

At both inner ($r = R$) and outer ($r = R + t$) surfaces, σ_{rr} should equal $-p$, i.e. in our case $\sigma_{rr}=0$. This gives us a system of two equations for constants a and b. Solving it, we get:

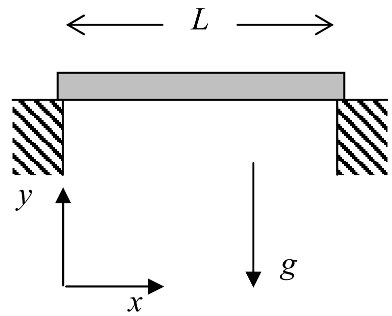
$$a=\frac{1}{8}C(3-2\sigma)\left[(R+t)^2+R^2\right]\approx\frac{1}{8}C(3-2\sigma)2R^2,$$
$$b=\frac{1}{8}C\frac{3-2\sigma}{1-2\sigma}(R+t)^2R^2\approx\frac{1}{8}C\frac{3-2\sigma}{1-2\sigma}R^4.$$

Plugging these values into Eq. (3), we find finally:

$$\Delta R \equiv s(R) = \frac{1}{8} CR^3 \left[(-1) + 2(3\sigma - 2) + \frac{3\sigma - 2}{1 - 2\sigma} \right] = \frac{\rho\Omega^2 R^3}{E}(1 - \sigma^2),$$

It is interesting that the deformation does not vanish in the limit $t \to 0$.

Problem 9.2. Calculate the deformation law $s_y(x)$ for a thin, heavy, elastic rod supported at both ends as shown in Figure below, and bent by its own weight. Compare the maximum rod deflection s_{max} with that calculated in class for the rod with one end clamped in the wall, assuming that the rod length L, its cross-section's "moment of inertia" I, Young's modulus E, and mass M are all the same.



Solution:

Just like for the problem solved in class, we need to integrate sequentially the system of four first-order differential equations (6.55)-(6.58). The problems differ only by the boundary conditions:

Case	Class		Home	
x	0	L	0	$L/2$
F		0		0
τ		0	0	
φ	0			0
s	0		0	

Using these conditions, we get the following results:

$$\text{Class: } s(x) = -\frac{\rho A g}{EI}\left(\frac{x^4}{24} - \frac{Lx^3}{6} + \frac{L^2x^2}{4}\right), \quad |s|_{\max} = \frac{\rho A g L^4}{8EI},$$

$$\text{Home: } s(x) = -\frac{\rho A g}{EI}\left(\frac{x^4}{24} - \frac{Lx^3}{12} + \frac{L^3x}{24}\right), \quad |s|_{\max} = \frac{5\rho A g L^4}{384EI},$$

Thus the maximum deflection in the homework case is almost ten times less than for the problem solved in class. This is natural since it is proportional to L^4, and the effective length of the beam for the homework problem is approximately twice less than for the class problem.