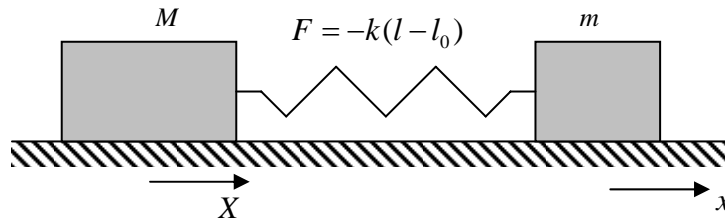


Midterm Exam 1 with Solutions

Problem M.1. Use the Lagrangian formalism to find the equations of frictionless motion of the system of two blocks coupled with a spring (see Figure below). Find the Hamiltonian function H and energy E . Are they equal? Is any of them conserved? Does the system have any other integral(s) of motion?



Solution: Using the horizontal positions X and x of the blocks as the generalized coordinates,¹ we have

$$T = \frac{M}{2} \dot{X}^2 + \frac{m}{2} \dot{x}^2, \quad U = \frac{k}{2} (X - x - l_0)^2.$$

Differentiating the Lagrange function $L = T - U$, we get the following equations of motion:

$$\begin{aligned} M\ddot{X} + k(X - x - l_0) &= 0, \\ m\ddot{x} - k(X - x - l_0) &= 0. \end{aligned} \tag{1}$$

Since $\partial L / \partial t = 0$, the system Hamiltonian H is conserved. Since T is a quadratic-homogeneous function of generalized velocities, system energy E

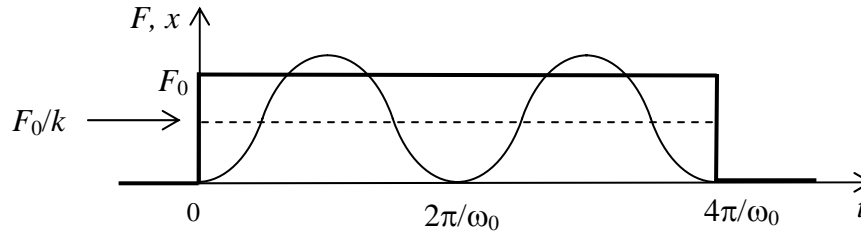
$$E = \frac{M}{2} \dot{X}^2 + \frac{m}{2} \dot{x}^2 + \frac{k}{2} (X - x - l_0)^2$$

is equal to H , and is also conserved. One more first integral of motion (total momentum) may be readily obtained by adding two equations (1) and integrating the result

$$M\dot{X} + m\dot{x} = \text{const.}$$

¹ Other choices of two generalized coordinates, e.g., X and $l = X - x$ are legitimate and of comparable convenience.

Problem M.2 A single square-wave pulse of force (see the Figure below) is exerted on a linear oscillator with own frequency ω_0 (no damping), initially at rest. Calculate the law of motion $x(t)$, sketch it, and interpret the result.



Solution: Following the Green's function method, we may express the solution as

$$x(t) = \int_0^{\infty} G(\tau) \frac{F(t-\tau)}{m} d\tau. \quad (1)$$

The Green's function of an oscillator has been calculated in class. In the limit of zero damping δ :

$$G(\tau) = \frac{\sin \omega_0 \tau}{\omega_0}.$$

Due to the piecewise-constant character of function $F(t)$, the non-vanishing parts of integral (1) have different limits (and hence give different final results) for the cases $0 < t < 4\pi/\omega_0$:

$$x(t) = \int_0^t G(\tau) \frac{F_0}{m} d\tau = \frac{F_0}{m\omega_0^2} (1 - \cos \omega_0 t),$$

and $t > 4\pi/\omega_0$:

$$x(t) = \int_{t-4\pi/\omega_0}^t G(\tau) \frac{F_0}{m} d\tau = 0.$$

The final result is sketched with a thin line in Figure above. Its physics is simple: the step of force applied at $t = 0$ shifts the equilibrium position from 0 to $\langle x \rangle = F_0 / m\omega_0^2 = F_0 / k$. Hence at $t = 0$ the oscillator is out of this new equilibrium point, and starts periodic oscillations around the new equilibrium position with amplitude $a = -\langle x \rangle = -F_0 / m\omega_0^2$. The equal and opposite step of force, arriving at time $t = 4\pi/\omega_0$, quenches these oscillations completely. (If the time interval between the two steps was not exactly a multiple of the oscillation period, the compensation would not be complete.)

Problem M.3. Find the fixed point of the following system of equations:

$$\dot{x} = -x - 4y,$$

$$\dot{y} = -y - x.$$

Analyze its stability. What type of the fixed point is it? (Node? saddle? focus? center?) Sketch the phase plane $[x,y]$ in as much detail as you can.

Solution: This system of equations is linear, so it has only one (trivial) fixed point $x = y = 0$, and linearization is unnecessary. Solving the characteristic equation of the system,

$$\begin{vmatrix} -1-\lambda & -4 \\ -1 & -1-\lambda \end{vmatrix} = 0,$$

we see that both characteristic exponents are real:

$$\lambda_{\pm} = -1 \pm 2. \quad (1)$$

One of them (λ_+) is positive and another negative, so that the fixed point is the (unstable) saddle.

Figure below shows several trajectories on the phase plane of this equation. The slopes of the separatrix and asymptote may be found by plugging the values of, respectively, λ_- and λ_+ back into the initial system of equations. This yields

$$y/x|_{\text{separatrix}} = 1/2, \quad y/x|_{\text{asymptote}} = -1/2,$$

in agreement with the plots.

