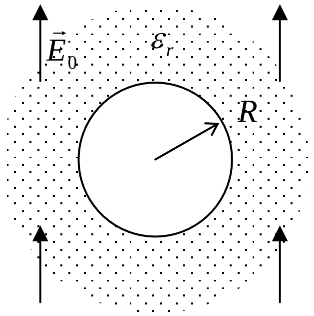
PHY 505 **Classical Electrodynamics I** Fall 2007

Final Exam with Solutions



Problem F.1 (150 points). A uniform electric field $\vec{E}_0$ has been created (by external sources) inside a uniform linear dielectric. Find the changes of fields $\vec{E}$, $\vec{D}$, and $\vec{P}$, created by cutting out a cavity in the shape of a round cylinder of radius *R*, with the symmetry axis perpendicular to the external field (see Fig. on the right).

Solution: Introducing the usual polar coordinates, we can use the general solution (2.103) of the Laplace equation as the guidance for looking for the electrostatic potential Φ in the following form:

$$\Phi\Big|_{\rho>R} = (-E_0\rho + \frac{b}{\rho})\cos\varphi,$$

$$\Phi\Big|_{\rho<R} = -E_{\text{int}}\rho\cos\varphi,$$

where (so far unknown) coefficient E_{int} has the sense of the uniform field inside the cavity. The boundary conditions of continuity of Φ and $\varepsilon\partial\Phi/\partial n$ (i.e. $\varepsilon\partial\Phi/\partial\rho$) on the cavity surface ($\rho = R$), we get two equations,

$$-E_0R + \frac{b}{R} = -E_{\text{int}}R,$$

$$\varepsilon_r\left(-E_0 - \frac{b}{R^2}\right) = -E_{\text{int}},$$

for two unknown coefficients, b and E_{int}. Solving them, we get:

$$b = -\frac{\varepsilon_r - 1}{\varepsilon_r + 1}E_0R^2, \quad E_{\text{int}} = \frac{2\varepsilon_r}{\varepsilon_r + 1}E_0.$$

As a result, the electrostatic potential distribution may be presented as

$$\Phi\Big|_{\rho>R} = -E_0\left(\rho + \frac{\varepsilon_r - 1}{\varepsilon_r + 1}\frac{R^2}{\rho}\right)\cos\varphi = -E_0x\left(1 + \frac{\varepsilon_r - 1}{\varepsilon_r + 1}\frac{R^2}{x^2 + y^2}\right),$$

$$\Phi\Big|_{\rho<R} = -\frac{2\varepsilon_r}{\varepsilon_r + 1}E_0\rho\cos\varphi = -\frac{2\varepsilon_r}{\varepsilon_r + 1}E_0x,$$

where x is the Cartesian coordinate along the initial field. As a (necessary:-) sanity check, at $\varepsilon_r = 1$ (uniform space with no dielectric), the potential distribution is the same at both $\rho > R$ and $\rho < R$:

$$\Phi = \Phi_0 = -E_0\rho\cos\theta,$$

and corresponds to the uniform field E_0. In the general case, $\varepsilon_r \neq 1$, the electric field, $\vec{E} = -\vec{\nabla}\Phi$, is:

$$\vec{E}\Big|_{\rho>R} = E_0\vec{n}_x + \frac{\varepsilon_r - 1}{\varepsilon_r + 1}E_0\left(\vec{n}_x\frac{R^2(y^2-x^2)}{\left(x^2+y^2\right)^2} - \vec{n}_y\frac{2R^2xy}{\left(x^2+y^2\right)^2}\right),$$

$$\vec{E}\Big|_{\rho<R} = \frac{2\varepsilon_r}{\varepsilon_r + 1}E_0\vec{n}_x.$$

where $\vec{n}_x$ is directed along the initial field. From here, the electric field change from its original value $E_0\vec{n}_x$ is:

$$\Delta\vec{E}\Big|_{\rho>R} = \frac{\varepsilon_r - 1}{\varepsilon_r + 1}E_0\left(\vec{n}_x\frac{R^2(y^2-x^2)}{\left(x^2+y^2\right)^2} - \vec{n}_y\frac{2R^2xy}{\left(x^2+y^2\right)^2}\right),$$

$$\Delta\vec{E}\Big|_{\rho<R} = \left(\frac{2\varepsilon_r}{\varepsilon_r + 1} - 1\right)E_0\vec{n}_x = \frac{\varepsilon_r - 1}{\varepsilon_r + 1}E_0\vec{n}_x.$$

Similarly, the changes of the fields $\vec{D} = \varepsilon\vec{E}$ and $\vec{P} = (\varepsilon - \varepsilon_0)\vec{E}$ from their initial values,

$$\vec{D}_0 = \varepsilon_0\varepsilon_r E_0\vec{n}_x \qquad \text{and} \qquad \vec{P}_0 = \varepsilon_0(\varepsilon_r - 1)E_0\vec{n}_x,$$

are

$$\Delta\vec{D}\Big|_{\rho>R} = \varepsilon_0\varepsilon_r\frac{\varepsilon_r - 1}{\varepsilon_r + 1}E_0\left(\vec{n}_x\frac{R^2(y^2-x^2)}{\left(x^2+y^2\right)^2} - \vec{n}_y\frac{2R^2xy}{\left(x^2+y^2\right)^2}\right),$$

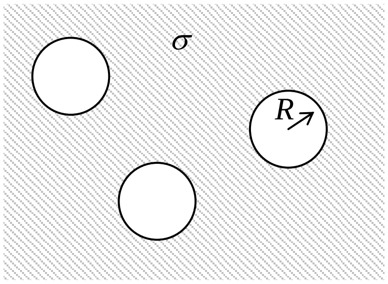
$$\Delta\vec{D}\Big|_{\rho<R} = \varepsilon_0\left(\frac{2\varepsilon_r}{\varepsilon_r + 1} - \varepsilon_r\right)E_0\vec{n}_x = -\varepsilon_0\varepsilon_r\frac{\varepsilon_r - 1}{\varepsilon_r + 1}E_0\vec{n}_x = -\varepsilon_0\varepsilon_r\Delta\vec{E}\Big|_{\rho<R},$$

and

$$\Delta\vec{P}\Big|_{\rho>R} = \varepsilon_0\frac{(\varepsilon_r - 1)^2}{\varepsilon_r + 1}E_0\left(\vec{n}_x\frac{R^2(y^2-x^2)}{x^2+y^2} - \vec{n}_y\frac{2R^2xy}{x^2+y^2}\right),$$

$$\Delta\vec{P}\Big|_{\rho<R} = \varepsilon_0\left[0 - (\varepsilon_r - 1)\right]E_0\vec{n}_x = -\varepsilon_0(\varepsilon_r - 1)E_0\vec{n}_x.$$

Notice that at $\rho < R$, $\Delta\vec{D} \neq \varepsilon\Delta\vec{E}$ and $\Delta\vec{P} \neq (\varepsilon - \varepsilon_0)\Delta\vec{E}$, because cutting out the cavity changes not only the field, but the electric permittivity as well.



Problem F.2 (200 points). Fid the effective (average) conductivity σ_{ef} of a medium with conductivity σ, with many empty spherical cavities of radius R (see Fig. on the right), in the limit of small sphere concentration, $n << R^{-3}$.

Hint: You may find the analysis of polarization in dielectrics, carried out in class, useful. However, since we are now dealing with conductors rather than dielectrics, please provide convincing arguments for all nontrivial steps of your calculations.

Solution: In class, we have found that a single spherical cavity, cut in a conductor with a uniform current flow of density $\vec{j}_0$, creates an additional dipole field

$$\Phi(r,\theta) = -\frac{j_0}{\sigma}\frac{R^3}{2r^2}\cos\theta .$$

According to Eq. (3.7), this field corresponds to the following dipole moment:

$$\vec{p} = -4\pi\varepsilon_0 \frac{R^3}{2\sigma}\vec{j}_0 .$$

If the concentration of spheres is low ($nR^3 << 1$), these dipole fields add up independently, creating the average electric polarization

$$\vec{P} = n\vec{p} = -4\pi\varepsilon_0 \frac{nR^3}{2\sigma}\vec{j}_0 .$$

Our analysis of electrically polarized media in Sec. 3.2 has shown that the polarization (divided by ε_0) is effectively subtracted from the electric field created by external sources. In the case of dielectrics, that initial field obeys the Poisson equation with external charge density ρ - see, e.g., the first term in the RHP of Eq. (3.28). In the case of a conductor, the initial field $\vec{E}_0$ is created by an external e.m.f., and equals $\vec{j}_0/\sigma$. However, all the arguments concerning the second term in that equation did not use any assumptions about the media, and may be used for conductors as well.

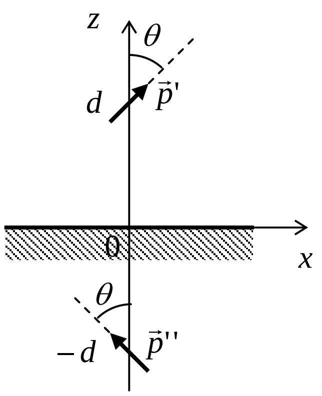
This means that in our case the average field is

$$\vec{E} = \vec{E}_0 - \frac{\vec{P}}{\varepsilon_0} = \frac{\vec{j}_0}{\sigma} + 4\pi\frac{nR^3}{2\sigma}\vec{j}_0 = \frac{\vec{j}_0}{\sigma_{\text{ef}}} ,$$

where σ_{ef} is the desirable effective conductance:

$$\sigma_{\text{ef}} = \frac{\sigma}{1 + 2\pi nR^3} \approx \sigma(1 - 2\pi nR^3) < \sigma .$$

The last equality is valid because our analysis is limited to the small cavity concentrations, $nR^3 << 1$, when the correction to σ is small. Of course, the reduction of conductivity, i.e. the sign of the difference $\Delta\sigma < 0$ due to carving out the cavities could be predicted from handwaving arguments, and so could be (less evidently) the proportionality $\Delta\sigma \propto nR^3$.



Problem F.3 (200 points). An electric dipole is located above an infinite conducting plane (see Fig. on the right). Find the distribution of the induced charge in the conductor.

Solution: This system has been considered in Homework Problem 6.2. We have seen that the field above the surface (in our Fig. , at z ≥ 0), may be presented as the sum of free-space fields of two dipoles, with the following coordinates,[1]

$$x' = x'' = 0, \quad y' = y'' = 0, \quad z' = -z'' = d\ ,$$

and dipole moment components:

$$p_x^{'} = -p_x^{''} = p\sin\theta, \quad p_y^{'} = p_y^{''} = 0, \quad p_z^{'} = p_z^{'''} = p\cos\theta.$$

According to Eq. (2.3), the required charge density may be calculated as

$$\sigma = \varepsilon_0 E_z\big|_{z=0}.$$

where $\vec{E}$ is the total field of the two dipoles. Since the vertical components of these dipole fields (at the observation point on the surface, $z = 0$, only!) are equal, we may use the well-known expression for the electric field of a dipole (see, e.g., Eq. (3.15) of the lecture notes) to write:

$$\sigma = 2\varepsilon_0\left[\frac{1}{4\pi\varepsilon_0}\frac{3(\vec{r}-\vec{r}\,')\big((\vec{r}-\vec{r}\,')\cdot\vec{p}\,'\big)-\vec{p}\,'(\vec{r}-\vec{r}\,')^2}{(\vec{r}-\vec{r}\,')^5}\right]_z = \frac{1}{2\pi}\frac{3(0-z')\big((\vec{r}-\vec{r}\,')\cdot\vec{p}\,'\big)-p_z^{'}(\vec{r}-\vec{r}\,')^2}{(\vec{r}-\vec{r}\,')^5}$$

$$= \frac{1}{2\pi}\frac{3(-d)\big(xp\sin\theta-dp\cos\theta\big)-p\cos\theta(x^2+y^2+d^2)}{(x^2+y^2+d^2)^{5/2}} = \frac{p}{2\pi}\frac{(2d^2-x^2-y^2)\cos\theta-3dx\sin\theta}{(x^2+y^2+d^2)^{5/2}}.$$

The structure of this expression is evidently close to that of the solution of Homework Problem 11.2.

Alternatively, the same result could be obtained as

$$\sigma = -\varepsilon_0\frac{\partial\Phi}{\partial z}\Big|_{z=0}\ ,$$

where Φ is the total potential of two dipoles for arbitrary *z*:

$$\Phi = \frac{1}{4\pi\varepsilon_0}\left[\frac{(\vec{r}-\vec{r}\,')\cdot\vec{p}\,'}{(\vec{r}-\vec{r}\,')^3}+\frac{(\vec{r}-\vec{r}\,'')\cdot\vec{p}\,''}{(\vec{r}-\vec{r}\,'')^3}\right] = \frac{p}{4\pi\varepsilon_0}\left[\frac{(z-d)\cos\theta+x\sin\theta}{\big(x^2+y^2+(z-d)^2\big)^{3/2}}+\frac{(z+d)\cos\theta-x\sin\theta}{\big(x^2+y^2+(z+d)^2\big)^{3/2}}\right].$$

However, the first way is simpler, because it does not require differentiation.

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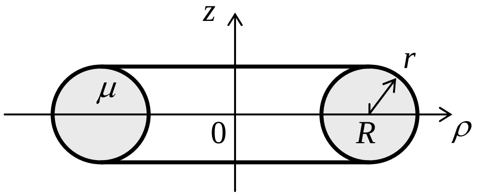
[1] The use of cylindrical coordinates would lead to much more complex calculations.

Problem F.4 (150 points). Calculate the (self-)inductance of a toroidal solenoid with the cross-section shown in Fig. on the right ($r \sim R$), filled with a material of magnetic permeability μ, with $N >> 1$ wire turns uniformly distributed along the perimeter. Check your results by analyzing the limit $r << R$.



Hint : You may like to use the following table integral:

$$\int_0^1 \ln\frac{a+\sqrt{1-\xi^2}}{a-\sqrt{1-\xi^2}}\,d\xi = \pi\left(a-\sqrt{a^2-1}\right), \quad \text{for } a \geq 1.$$

Solution: In class, we have calculated magnetic field inside such a solenoid without magnetic filling – see Eq. (5.) Since all the field is concentrated inside the solenoid, if it is filled with a magnetic material, we can just multiply this result by the ratio μ/μ_0:

$$B = \frac{\mu NI}{2\pi\rho}.$$

We can now calculate the magnetic flux piercing one wire loop:

$$\Phi_1 = \int_A B_n d^2r = \frac{\mu NI}{\pi}\int_0^r dz \int_{R-\sqrt{r^2-z^2}}^{R+\sqrt{r^2-z^2}} \frac{d\rho}{\rho} = \frac{\mu NI}{\pi}\int_0^r \ln\frac{R+\sqrt{r^2-z^2}}{R-\sqrt{r^2-z^2}}\,dz$$

$$= \frac{\mu NIr}{\pi}\int_0^1 \ln\frac{R/r+\sqrt{1-\xi^2}}{R/r-\sqrt{1-\xi^2}}\,d\xi = \mu NI\left(R-\sqrt{R^2-r^2}\right).$$

Just as for the long solenoid discussed in class, the flux Φ piercing the whole wire is N times larger. As a result, the solenoid inductance

$$L = \frac{\Phi}{I} = \mu N^2\left(R-\sqrt{R^2-r^2}\right).$$

In the limit $r << R$, we may expand this expression into the Taylor series in small r/R, and in the first approximation get

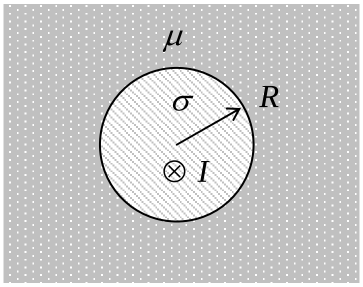
$$L \approx \mu N^2\frac{r^2}{2R} = \mu n^2 lA, \quad \text{with } n \equiv \frac{N}{l}, \quad l \equiv 2\pi R, \quad \text{and } A \equiv \pi r^2.$$

We see that in this limit, the result coincides with the inductance of a long straight solenoid, calculated in class – see Eq. (5.82).

It is interesting that in the opposite limit ($r = R$), our result for the inductance also acquires a very simple form,

$$L = \mu N^2 r\,.$$

Notice that it may be used for relatively small values of N (say, $N = 1$) only if $\mu >> \mu_0$, otherwise we have to account for "stray" magnetic fields spilling out of the torus interior.



Problem F.5 (100 points). DC current I is being passed through a long, straight conductor with a round cross-section of radius R, constant conductivity σ, and negligible magnetic susceptibility ($\mu = \mu_0$). The conductor is embedded in an infinite, uniform, non-conducting magnetic medium with $\mu \neq \mu_0$ (see Fig. on the right). Find the distribution of magnetic fields $\vec{B}$ and $\vec{H}$ everywhere in the system.

Solution: Due to the constant conductivity, current is distributed uniformly over the conductor cross-section area. Due to the axial symmetry of the system, magnetic lines of both fields ($\vec{B}$ and $\vec{H}$) form rings concentric with the wire. Applying the generalized Ampere law,

$$\oint_C \vec{H} \cdot d\vec{l} = \int_{A(\rho)} j_n d^2 r \, ,$$

to a circular contour of radius ρ, we get

$$H = \frac{I}{2\pi} \begin{cases} \rho / R^2, & \text{for } \rho \leq R, \\ 1/\rho, & \text{for } \rho \geq R. \end{cases}$$

As a result, for field $\vec{B}$ we get

$$B = \frac{I}{2\pi} \begin{cases} \mu_0 \rho / R^2, & \text{for } \rho \leq R, \\ \mu / \rho, & \text{for } \rho \geq R. \end{cases}$$

It is easy to check that all the relevant boundary conditions at the interface $\rho = R$ are satisfied: H_τ = const, j_n = const, and B_n = const (in our case $j_n = B_n = 0$).