

Problem 1.1. (10 points.) Use any two (different) approaches you like to calculate the distribution of electrostatic potential Φ and electric field \vec{E} created by a plane layer of thickness t , with a uniformly distributed charge of volume density ρ (see Fig.).



Solution 1: Using as the Gauss volume a pillbox similar to that discussed in class (Fig. 1.3), for $|z| > t/2$ we get the same field as for the thin charged plane - see Eq. (24), where now $\sigma = t\rho$, so that

$$E = \frac{\rho}{\epsilon_0} \frac{t}{2}, \quad E_z = \frac{\rho}{\epsilon_0} \frac{t}{2} \text{sgn}(z), \quad \text{for } |z| > \frac{t}{2}.$$

On the other hand, if the pillbox thickness is smaller than the charged layer thickness, then the areal density participating in the Gauss Law is only that inside the pillbox, so that $\sigma = 2|z|\rho$, so that the law yields

$$E = \frac{\rho}{\epsilon_0} |z|, \quad E_z = \frac{\rho}{\epsilon_0} z, \quad \text{for } |z| < \frac{t}{2}$$

The formulas (of course) give the same result on the layer boundaries.

Now, the electrostatic potential may be obtained by integration of E , for example, along the vertical axis:

$$\Phi = -\int E_z dz = \Phi_{|z=0} - \int_0^z E(z') dz' = \Phi_{|z=0} - \frac{\rho}{2\epsilon_0} \times \begin{cases} z^2, & \text{for } |z| < t/2, \\ t\left(|z| - \frac{t}{4}\right), & \text{for } |z| > t/2, \end{cases} \quad (*)$$

where the value on the symmetry plane is arbitrary. Note if the charge per unit area, $\sigma = t\rho$, is fixed, the electric field outside the layer does not depend on its thickness, but the electric potential does, reflecting the additional energy of the field inside the layer due to its finite thickness.

Another notable feature of result (*) result is that the electrostatic potential grows infinitely as $z \rightarrow \pm\infty$. This is of course an artifact of our (unphysical) assumption that the charged sheet's area is infinite. For any realistic (finite-area) sheet the electric field and the potential start to fall as soon as $|z|$ becomes larger than the sheet size.

Solution 2: The Poisson equation for this system takes the form

$$\nabla^2 \Phi = -\begin{cases} \rho/\epsilon_0, & \text{for } |z| < t/2, \\ 0, & \text{for } |z| > t/2. \end{cases}$$

It is evidently satisfied by a 1D solution $\Phi = \Phi(z)$, so that the equation takes the form:

$$\frac{d^2 \Phi}{dz^2} = -\begin{cases} \rho/\epsilon_0, & \text{for } |z| < t/2, \\ 0, & \text{for } |z| > t/2. \end{cases}$$

Integrating the former (top) equation once, we get

$$\frac{d\Phi}{dz} = -\frac{\rho}{\epsilon_0} z + \text{const.}$$

The integration constant may be determined from the problem symmetry: $\Phi(-z) = \Phi(z)$. Near $z = 0$, this equation gives

$$\left. \frac{d\Phi}{dz} \right|_{z=0} = 0,$$

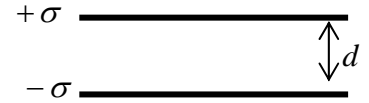
showing that the integration constant should equal zero. Now we can integrate the result over z again, arriving at the top line of Eq. (*). The integration of the Poisson for region $|z| > d/2$ is similar, with the boundary conditions provided by the continuity of functions $\Phi(z)$ and $d\Phi/dz = -E_z(z)$ at the interfaces $|z| > d/2$. The result is given by the lower line of Eq. (*).

We see that the second method of solution (based on the Poisson equation) is somewhat less convenient than using the Gauss Law. As a partial compensation, the intermediate result, $d\Phi/dz$, already gives the electric field E_z (with the minus sign).

Problem 1.2. (10 points) Find:

- (i) the distribution of Φ and \vec{E} , and
- (ii) the electrostatic energy per unit area

of two thin, parallel planes with equal and opposite charges of constant areal density σ , and separated by distance d (see Fig.).



Solution: From the similar problem solved in class (see Fig. 1.3 and its discussion), it is clear that the electric field everywhere is vertical and constant within each of three ranges:

- (i) below the lower plane ($E_z = E_-$),
- (ii) between the planes ($E_z = E_0$), and
- (iii) above the top plate ($E_z = E_+$).

Applying the Gauss Law to three pillboxes with three possible combinations of the lid positions, we get three equations for these constants:

$$E_+ - E_0 = +\sigma/\epsilon_0, \quad E_0 - E_- = -\sigma/\epsilon_0, \quad E_+ - E_- = 0,$$

which give $E_+ = E_- = 0$, $E_0 = -\sigma/\epsilon_0$. Thus, the field exists only between the planes, producing the electrostatic potential

$$\Phi = \int \Phi|_{z=0} - \int_0^z E(z') dz' = \Phi|_{z=0} + \frac{\sigma}{\epsilon_0} \times \begin{cases} z, & \text{for } |z| < d/2, \\ (d/2) \text{sgn}(z), & \text{for } |z| > d/2, \end{cases}$$

so that the potential difference (“voltage”) between the planes is

$$V \equiv \Phi\left(+\frac{d}{2}\right) - \Phi\left(-\frac{d}{2}\right) = \frac{\sigma}{\epsilon_0} d \equiv \frac{\sigma}{C_0},$$

where $C_0 = \varepsilon_0/d$ is the “specific capacitance” of this simple system (which is equivalent to a *plane capacitor* – see Ch. 2). The easiest way to calculate the electrostatic energy per unit area is to use Eq. (1.53):

$$\frac{U}{A} = \frac{\varepsilon_0}{2} \int_{-d/2}^{+d/2} E^2 dz = \frac{\varepsilon_0 E_0^2}{2} d = \frac{\sigma^2}{2\varepsilon_0} d = \frac{C_0 V^2}{2}.$$

Problem 1.3. (10 points) Use the Poisson equation to find the electrostatic potential Φ created by a sphere of radius R , whose volume is charged with constant density ρ . From that result, calculate the electric field distribution and compare it with the formulas derived in class from the Gauss Law.

Solution: Using the problem symmetry, $\Phi(\vec{r}) = \Phi(r)$, we can use the following simple expression for the Laplace operator:

$$\nabla^2 \Phi = \frac{1}{r^2} (r^2 \Phi'),$$

so that for the points inside the charged sphere ($r < R$) the Poisson equation yields

$$\frac{1}{r^2} (r^2 \Phi') = -\frac{\rho}{\varepsilon_0}.$$

Integrating it once, we have

$$\Phi'(r) = -\frac{\rho}{r^2 \varepsilon_0} \int_0^r r'^2 dr' = -\frac{\rho r}{3\varepsilon_0} = -\frac{1}{4\pi\varepsilon_0} \frac{Qr}{R^3}. \quad (*)$$

Since this derivative is nothing more than $-E(r)$, we can readily recognize in this formula our previous result (1.22). We can now carry out the second integration to calculate the potential itself:

$$\Phi(r) = -\frac{Q}{4\pi\varepsilon_0 R^3} \int_0^r r' dr' + C_1 = -\frac{Qr^2}{8\pi\varepsilon_0 R^3} + C_1.$$

Before making any judgment on the integration constant C_1 , let us solve the Poisson equation for the range outside the sphere ($r > R$):

$$\frac{1}{r^2} (r^2 \Phi') = 0.$$

Its first integral,

$$\Phi'(r) = \frac{C_2}{r^2},$$

gives the electric field (with the minus sign). Using Eq. (*) and requiring the field to be continuous at $r = R$, we get

$$\frac{C_2}{R^2} = -\frac{Q}{4\pi\varepsilon_0 R^2}, \quad \text{i.e. } \Phi'(r) = -\frac{Q}{4\pi\varepsilon_0 r^2},$$

in an evident agreement with Eq. (1.19). Integrating this result again,

$$\Phi(r) = -\frac{Q}{4\pi\epsilon_0} \int \frac{dr}{r^2} = \frac{Q}{4\pi\epsilon_0 r} + C_3, \quad \text{for } r > R, \quad (***)$$

we can select $C_3 = 0$, so that $\Phi(\infty) = 0$, in accordance with the usual convention.

Now we can finally determine constant C_1 in Eq. (**) by requiring that this equation and Eq. (***) give the same value of Φ on the boundary $r = R$. (If the potential had a jump, electric field in that point would be infinite.) The final answer may be presented as

$$\Phi(r) = \frac{Q}{4\pi\epsilon_0 R} \left[\frac{R^2 - r^2}{2R^2} + 1 \right], \quad \text{for } r \leq R.$$

We see that for cases like this the Poisson equation approach is less convenient than using the Gauss Law. However, this equation may be indispensable in the cases when the charge distribution has to be calculated in self-consistent way, in parallel with the electric potential it creates.

Problem 1.4. (10 points) Calculate the electrostatic energy of the uniformly charged sphere, using any two (different) methods. (You are welcome to use the results of the Problem 1.3 if you like.)

Solution 1: Using Eq. (1.47), we may limit integration by the sphere volume ($0 \leq r \leq R$) where $\rho \neq 0$. Using the last equation in the solution of Exercise 1.1, we get

$$U = \frac{1}{2} 4\pi \int_0^R \rho \Phi r^2 dr = \frac{1}{2} 4\pi \rho \frac{Q}{4\pi\epsilon_0 R} \int_0^R \left[\frac{R^2 - r^2}{2R^2} + 1 \right] r^2 dr = \frac{6}{5} \frac{1}{4\pi\epsilon_0 R} \frac{Q^2}{2}. \quad (*)$$

Solution 2: Alternatively, one can use Eq. (1.53), but in this case we need to integrate energy anywhere where electric field is, i.e. both inside and outside of the sphere:

$$U = \frac{\epsilon_0}{2} 4\pi \left[\int_0^R E^2 r^2 dr + \int_R^\infty E^2 r^2 dr \right].$$

Using Eqs. (1.19) and (1.22) for, respectively, the external and internal regions, we obtain:

$$U = \frac{\epsilon_0}{2} 4\pi \left[\int_0^R \left(\frac{Qr}{4\pi\epsilon_0} \right)^2 r^2 dr + \int_R^\infty \left(\frac{Q}{4\pi\epsilon_0 r^2} \right)^2 r^2 dr \right] = \left(\frac{1}{5} + 1 \right) \frac{1}{4\pi\epsilon_0 R} \frac{Q^2}{2}.$$

This is (fortunately :-)) the same answer as given by Eq. (*), but to some extent it is more informative because it shows how exactly the electric field energy is distributed between the interior and exterior of the charged sphere. We will refer to these results in Ch. 2.