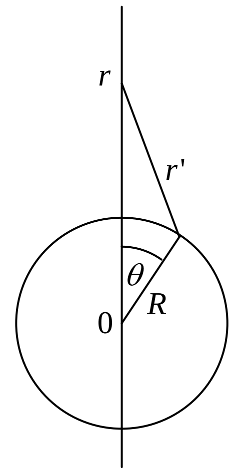
PHY 505 **Classical Electrodynamics I** Fall 2007

Homework 02 with Solutions

Problem 2.1 (10 points). Calculate the average electric potential of the spherical surface of radius R, created by a point charge q located at distance $r > R$ from the sphere's center.



Solution: Using the evident axial symmetry of the problem (see Fig.), we get:

$$\Phi_{\text{ave}} \equiv \frac{1}{4\pi}\oint \Phi(\theta)d\Omega = \frac{2\pi}{4\pi}\int_0^{\pi}\Phi(\theta)\sin\theta d\theta = \frac{1}{2}\int_0^{\pi}\frac{q}{4\pi\varepsilon_0 r'}\sin\theta d\theta,$$

where r' is the distance between the point charge and the observation point:

$$(r')^2 = R^2 + r^2 - 2Rr\cos\theta\,.$$

The integral may be readily taken via the transfer to a new variable $\xi \equiv \cos\theta$ (so that $\sin\theta\, d\theta = d\xi$):

$$\Phi_{\text{ave}} = \frac{1}{2}\frac{q}{4\pi\varepsilon_0}\int_1^{-1}\frac{d\xi}{\left[R^2 + r^2 - 2Rr\xi\right]^{1/2}} = \frac{1}{2}\frac{q}{4\pi\varepsilon_0 r'}\frac{2}{(-2Rr)}\left[R^2 + r^2 - 2Rr\xi\right]^{1/2}\Big|_{\xi=-1}^{\xi=+1}$$

$$= \frac{1}{2}\frac{q}{4\pi\varepsilon_0}\frac{2}{(-2Rr)}\left\{\left[R^2 + r^2 - 2Rr\right]^{1/2} - \left[R^2 + r^2 + 2Rr\right]^{1/2}\right\} = \frac{q}{4\pi\varepsilon_0 r}.$$

We see that the average coincides with the potential value in the middle of the sphere. (Notice that this result is only valid for the case $r > R$.

Problem 2.2 (5 points). Use the result of the previous problem to prove the following general *mean value theorem*: The value of the electric potential in any point is equal to its average value on the surface of any sphere with the center in that point and containing no charges inside it.

Solution: The proof is elementary using the linear superposition principle: since the relation

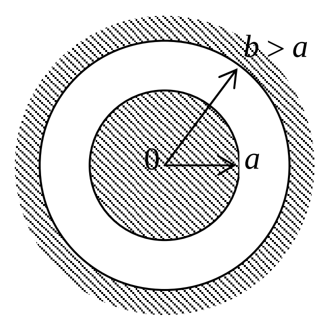
$$\frac{q}{4\pi\varepsilon_0 r} = \Phi_{\text{ave}}$$

holds for each point charge located outside the sphere (as was proved in the previous problem), it is also true for any system of such charges.

Problem 2.3 (10 points). Use the "macroscopic" boundary conditions to calculate the mutual capacitance of the following 2-electrode systems:

(i) a conducting sphere inside a concentric spherical cavity in another conductor (see Fig.), and

(ii) a conducting cylinder inside a coaxial cavity in another conductor.



(In the latter case, we speak about the conductance per unit length.) Analyze the results and compare them to the capacitance between two parallel conducting planes.

Solutions:

(i) Applying the Gauss Law to a sphere of radius r in the range $a < r < b$, for the electric field $\vec{E} = E(r)\vec{n}_r$ we get

$$E(r) = \frac{Q}{4\pi\varepsilon_0 r^2},$$

where Q is the charge of the inner conductor. Integrating this result, we find that voltage $V \equiv \Phi(b) - \Phi(a)$ between the electrodes is

$$V = \int_a^b E(r)dr = \frac{Q}{4\pi\varepsilon_0}\int_a^b \frac{dr}{r^2} = \frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{a} - \frac{1}{b}\right),$$

so that the mutual capacitance

$$C_m \equiv \frac{Q}{V} = 4\pi\varepsilon_0\left(\frac{1}{a} - \frac{1}{b}\right)^{-1} = 4\pi\varepsilon_0 \frac{ab}{b-a}.$$

In the limit $a << b$, the result approaches $4\pi\varepsilon_0 a$, i.e. the self-capacitance of a sphere of radius a. On the other hand, if $a \to b$, i.e. the gap between two conductors is narrow,

$$C_m \to 4\pi\varepsilon_0 \frac{a^2}{b-a} = \frac{\varepsilon_0 A}{d}, \quad \text{with } d \equiv b - a, \;\; A \equiv 4\pi a^2 \approx 4\pi b^2,$$

i.e. the capacitance approaches that of the planar capacitor or area A (as it should).

(ii) An absolutely similar calculation, but applied to a cylinder of unit length rather than a sphere, gives (cf. Exercise 1.5)

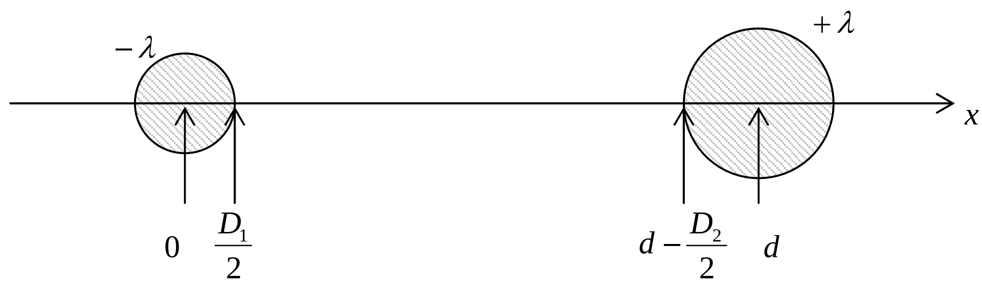
$$E(r) = \frac{\lambda}{2\pi\varepsilon_0 r}, \qquad \frac{C_m}{L} = \frac{2\pi\varepsilon_0}{\ln(b/a)},$$

where $\lambda \equiv Q/L$ is the charge per unit length. At $a \to b$, $\ln(b/a) \to (b - a)/a$, so that the capacitance

$$\frac{C_m}{L} \to \frac{\varepsilon_0 A}{d}, \quad \textit{with } d = b - a, \;\; A = 2\pi aL \approx 2\pi bL,$$

is again approaches that of a plane capacitor. However, in contrast with the case of the sphere, in the limit $a/b \to 0$ the capacitance diverges, albeit very weakly (logarithmically).

Problem 2.4 (5 points). Following the class discussion of two weakly coupled spheres, find an approximate expression for the mutual capacitance (per unit length) between two thin, parallel wires, each with a round cross-section, but its own diameter. Compare the result with that for two spheres and interpret the difference.



Solution: This problem is close to that solved in class for two small spheres, but we should be more accurate with the selection of the constant in the electrostatic potential, because the potential of a single, uniformly charged wire is proportional to ln r and hence diverges at $r \to \infty$. As a result, the *self-capacitance* of a single wire of an infinite length is not well defined – see, e.g., Footnote 8 in the lecture notes. This is why it is better to immediately proceed to the calculation of the *mutual* capacitance between two wires by considering them uniformly charged with equal but opposite linear densities $\pm\lambda$ - see Fig. above.

Just as in the problem solved in class, when calculating the electric field E_1 created by the left wire (charge $-\lambda$) alone, we can neglect the effects of the second wire. Applying the Gauss Law to a round cylinder of radius r, coaxial with the first wire, we get (just like in Problem 2.3b above):

$$E_{1r} = -\frac{\lambda}{2\pi\varepsilon_0 r},$$

so that the electric field along the line connecting the wire centers is

$$E_{1x} = -\frac{\lambda}{2\pi\varepsilon_0 x}.$$

Similarly, with our choice of the origin of axis x (see Fig. above), the field created on that line by the second wire is

$$E_{2x} = -\frac{\lambda}{2\pi\varepsilon_0 (d-x)}.$$

(Here we took into account the opposite sign of the distributed charge.)

Now we can get calculate voltage V between the wires by integration of the total field $E_x = E_{1x} + E_{2x}$ along axis x:

$$V = \Phi_2 - \Phi_1 = \Phi\Big|_{x=d-D_2/2} - \Phi\Big|_{x=D_1/2} = -\int_{D_1/2}^{d-D_2/2}(E_{1x}+E_{2x})dx = \frac{\lambda}{2\pi\varepsilon_0}\int_{D_1/2}^{d-D_2/2}\left(\frac{1}{x}+\frac{1}{d-x}\right)dx$$

$$= \frac{\lambda}{2\pi\varepsilon_0}\left(\ln\frac{d-D_2/2}{D_1/2} - \ln\frac{D_2/2}{d-D_1/2}\right) \approx \frac{\lambda}{2\pi\varepsilon_0}\ln\frac{4d^2}{D_1 D_2},$$

where the last transition is based on the strong inequality $d >> D_{1,2}$. Hence the mutual capacitance per unit length

$$\frac{C_m}{L} \equiv \frac{\lambda}{V} \approx \frac{2\pi\varepsilon_0}{\ln\left(\dfrac{4d^2}{D_1 D_2}\right)} = \frac{\pi\varepsilon_0}{\ln\left(d / R_{\text{mean}}\right)}, \quad \text{where } R_{\text{mean}}^2 \equiv \frac{D_1}{2}\frac{D_2}{2}.$$

This result shows, that, in contrast with the similar problem solved in class, the mutual capacitance does depend on the distance between the conductors (decreasing with d), albeit very weakly (logarithmically). This a result of the logarithmic divergence of the single wire's potential, mentioned above.

Also notice the close similarity between the last formula and that for the "coaxial cable" geometry considered in Problem 2.3b.