PHY 505 **Classical Electrodynamics I** Fall 2007

Homework 05 with Solutions

Problem 5.1 (10 points). Use the variable separation method to find the potential distribution

(i) outside, and

(ii) inside

a thin spherical shell of radius R, whose surface has potential $\Phi(R,\theta,\varphi) = V_0 \sin\theta\cos\varphi$.

Solution: The surface potential corresponds to a linear combination of just two spherical harmonics $Y_l^n(\theta,\varphi)$, namely $Y_1^1 \propto \sin\theta \exp\{+i\varphi\}$ and $Y_1^{-1} \propto \sin\theta \exp\{-i\varphi\}$. Also, inside the sphere we can use only radial functions $A_l r^l$ which do not diverge at $r \to 0$; while for the outer problem we may only use functions B_l/r^{l+1} which do not diverge at $r \to \infty$. As a result, the general solution to the Laplace equation in spherical coordinates is reduced to

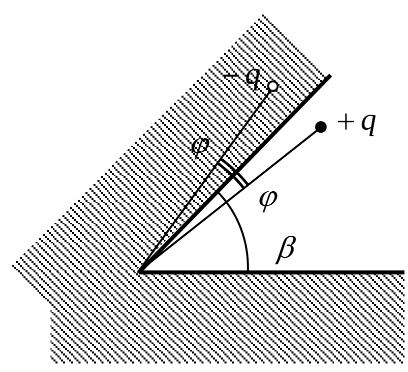
$$\Phi(r,\theta,\varphi) = \begin{cases} A_1 r \sin\theta\cos\varphi, & \text{for } r < R, \\ \dfrac{B_1}{r^2}\sin\theta\cos\varphi, & \text{for } r > R. \end{cases}$$

Finding constants A_1 and B_1 from the boundary condition on the spherical shell ($r = R$), we get finally

$$\Phi(r,\theta,\varphi) = V_0 \begin{cases} (r/R)\sin\theta\cos\varphi, & \text{for } r < R, \\ (R/r)^2 \sin\theta\cos\varphi, & \text{for } r > R. \end{cases}$$

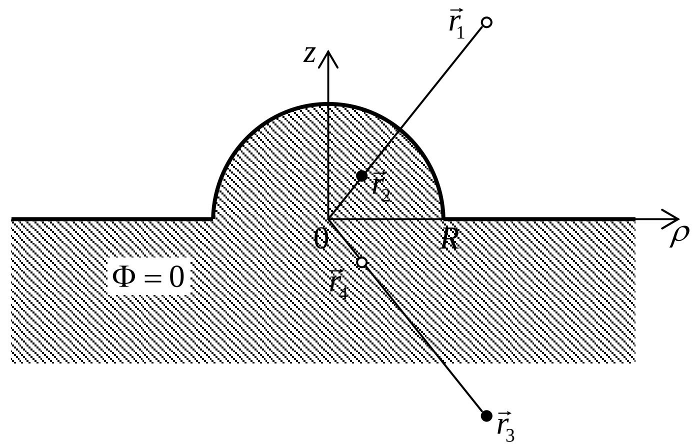
Note that the outer potential has the spatial dependence ($\propto 1/r^2$) of a dipole. Additional question for a (modest:-) additional credit: what is the dipole moment $\vec{p}$ of the system (both magnitude and direction)?

Problem 5.2 (5 points). For what values of the corner angle β, the boundary problem, shown in Fig. on the right, can be solved using a finite number of image charges? (Justify your answer.)



Solution: Let the angle between the direction to the charge and the nearest wall equal φ (see Fig.), so that its angular distance from the opposite wall is (β - φ), The reflection of the charge in the nearest wall will give a negative image charge in the direction ($\beta + \varphi$). Now the reflection of that pair in the opposite wall will give charge $+q$ at angles ($-\beta$ - φ) and charge $-q$ at ($-\beta + \varphi$). We see that the angular distance between the pairs is 2β. Repeating this charge pair reflection procedure again and again (say, n times total), we get n pairs of the same polarity, with centers within the sector $2n\beta$ wide. In order to have the process stopped, we need the last generated pairs to overlap, so that $2n\beta$ should equal 2π, i.e. $\beta = \pi/n$, with $n = 1, 2, \ldots$

Problem 5.3 (10 points). Use the method of images to find the Green's function of the system shown in Fig. on the right. (The bulge on the conducting plane has the shape of a semi-sphere of radius R.)



Solution: Let a (real) point charge q_1 be at point $\vec{r}_1$. Then all the boundary conditions may be satisfied using three charge images (see Fig.), with

$$q_2 = -q_1 \frac{R}{r_1}, \quad \vec{r}_2 = \left(\frac{R}{r_1}\right)^2 \vec{r}_1,$$
$$q_3 = -q_1, \quad \vec{\rho}_3 = \vec{\rho}_1, \quad z_3 = -z_1,$$
$$q_4 = -q_3 \frac{R}{r_3}, \quad \vec{r}_4 = \left(\frac{R}{r_3}\right)^2 \vec{r}_3,$$

where $\vec{\rho}_j$ is the horizontal component of radius-vector $\vec{r}_j$. As a result, the Green's function may be presented as

$$G(\vec{r}, \vec{r}_1) = \sum_{j=1}^{4} \frac{q_j / q_1}{\left|\vec{r} - \vec{r}_j\right|}.$$

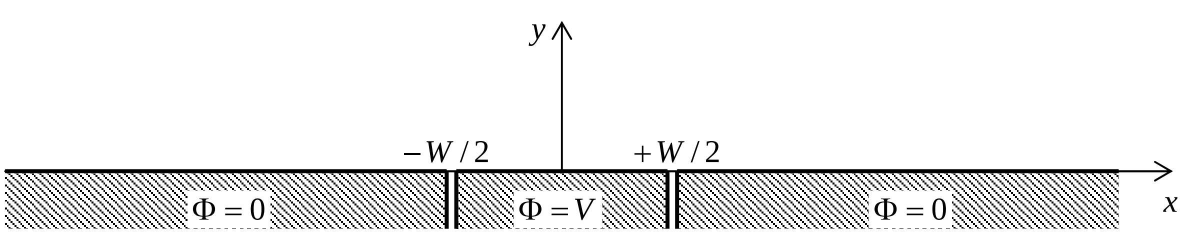
Problem 5.4 (15 points). Find the 2D Green's function in:

(i) unlimited free space, and

(ii) free space above a conducting plane.

Use the latter result to calculate the distribution of the electric potential created by the 2D system shown in Fig. below. (The gaps between the conducting fragments are negligibly small.)

Hint: Physically, the 2D Green's function may be viewed as the electrostatic potential created by a thin, straight, infinitely long wire, with the uniform charge density $\lambda = 4\pi\varepsilon_0$.



Solutions:

(i) In the unlimited free space, we may use the Gauss law to find the electric field

$$E_r = \frac{\lambda}{2\pi\varepsilon_0 \rho},$$

and from here the electrostatic potential

$$\Phi = -\int E_r d\rho = -\frac{\lambda}{2\pi\varepsilon_0} \ln \rho + const$$

generated by a thin line of charges. (Here ρ is the shortest distance from the field observation point to the line of charges.) Hence the 2D Green's function may be presented as

$$G(\vec{\rho},\vec{\rho}') = -2\ln|\vec{\rho}-\vec{\rho}'| + \text{const} = -\ln|\vec{\rho}-\vec{\rho}'|^2 + \text{const},$$

where $\vec{\rho}$ and $\vec{\rho}'$ are the 2D radius-vectors of the observation point and the charged line trace ("cross-section"), respectively.

(ii) For a charged line parallel to a grounded conducting plane $y = 0$, crossing the plane of drawing in point $\vec{\rho} = \{x', y'\}$, both the Poisson equation and boundary condition, $\Phi(x,0) = 0$, may be satisfied by the introduction of a image line with charge density $-\lambda$, crossing the plane of drawing in point $\vec{\rho}'' = \{x', -y'\}$. Hence the Green's function of the system is

$$G(\vec{\rho},\vec{\rho}') = -\ln|\vec{\rho}-\vec{\rho}'|^2 + \ln|\vec{\rho}-\vec{\rho}''|^2 = -\ln\left[(x-x')^2 + (y-y')^2\right] + \ln\left[(x-x')^2 + (y+y')^2\right]. \quad (*)$$

(iii) What we need for applications is the normal component of the gradient of the Green's function at the surface. A straightforward differentiation of Eq. (4) yields

$$\frac{\partial G}{\partial n'}\Big|_A = \frac{\partial G}{\partial y'}\Big|_{y'=0} = \frac{4y}{(x-x')^2 + y^2}.$$

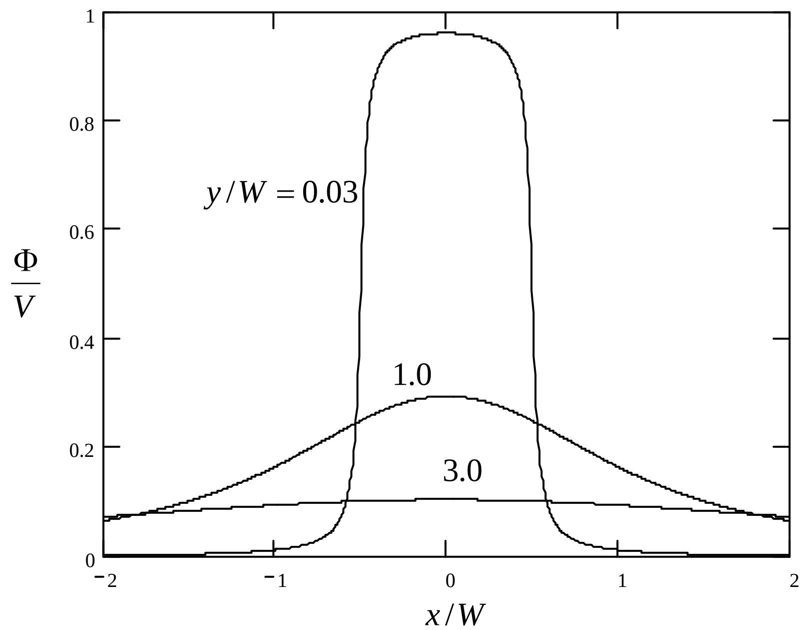
Plugging this expression into the 2D version of the main formula of the Green's function theory (with zero free charge), we get

$$\Phi(\vec{\rho}) = \frac{1}{4\pi}\sum_k \Phi_k \oint_{L_k} \frac{\partial G(\vec{\rho},\vec{\rho}')}{\partial n'} dl' = \frac{V}{4\pi}\int_{-W/2}^{+W/2} \frac{4y}{(x-x')^2 + y^2} dx'.$$

This integral may be readily taken using the substitution $\xi \equiv (x - x')/y$, giving

$$\Phi = -\frac{V}{\pi}\int_{(x+W/2)/y}^{(x-W/2)/y} \frac{d\xi}{\xi^2+1} = \frac{V}{\pi}\left[\arctan\frac{x+W/2}{y} - \arctan\frac{x-W/2}{y}\right].$$

As the numerical plot of this result (see Fig. below) shows, it result describes a "bump"-shaped distribution of the potential along axis x. Close to the conducting plane, the bump is sharp, and its height approaches V, but at large distances from the surface the bump is lower and more smooth, with width $\Delta x \sim y$.



1
0.8
0.6
0.4
0.2
0
Φ/V
y/W = 0.03
1.0
3.0
−2
−1
0
1
2
x/W