PHY 505 **Classical Electrodynamics I** Fall 2007

Homework 06 with Solutions

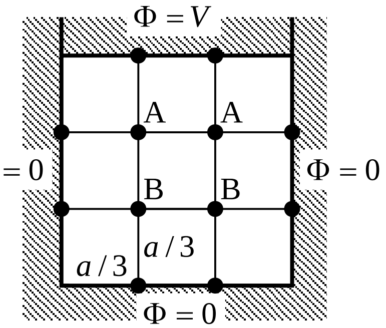
Problem 6.1 (20 points). Solve the same 2D boundary problem which has been discussed in class (see Fig. on the right) using:

(i) the finite difference method, with a finer square mesh, $h = a/3$, and

(ii) the variable separation method.

Compare the results (in the mesh points only) and comment.

Hint: Use the problem symmetry to reduce the number of unknown potentials.



Solutions:

(i) Due the problem symmetry, there are only two types of internal nodes in this mesh: A and B (see Fig.), so we need only two finite difference equations, each describing a 5-point “cross” with the center in each of these points:

- point A:

$$0 + \Phi_\mathrm{A} + V + \Phi_\mathrm{B} - 4\Phi_\mathrm{A} = 0,$$

- point B:

$$0 + \Phi_\mathrm{B} + \Phi_\mathrm{A} + 0 - 4\Phi_\mathrm{B} = 0.$$

Solving this simple system of equations, we get

$$\Phi_\mathrm{A} = \frac{3}{8}V = 0.375\,V, \quad \Phi_\mathrm{B} = \frac{1}{8} = 0.125\,V.$$

(ii) The solution of this 2D boundary Laplace problem using the variable separation is (see, e.g., model solution of Midterm Problem M.3):

$$\Phi(x,y) = \sum_{n=1}^{\infty} C_n \sin\frac{\pi n x}{a} \sinh\frac{\pi n y}{a},$$

where coefficients C_n have to be found from the boundary condition on the top lid ($y = a$):

$$V = \sum_{n=0}^{\infty} C_n \sin\frac{\pi n x}{a} \sinh \pi n.$$

Performing the reciprocal Fourier transform (i.e., multiplying both parts of this equation by $\sin(\pi n' x/a)$, and integrating over x from 0 to a), we get

$$C_n = \frac{1}{(a/2)\sinh \pi n} V \int_0^a \sin\frac{\pi n x}{a} dx = \begin{cases} (4V/\pi n)\sinh \pi n, & \text{for } n = 2j+1, \\ 0, & \text{for } n = 2j, \end{cases} \qquad j = 0, 1, 2...$$

For the potentials in the $a/3$ mesh nodes, these equations yield

$$\Phi_\mathrm{A} = \frac{4}{\pi} V \sum_{j=0}^{\infty} \sin\frac{\pi(2j+1)}{3} \frac{\sinh[2\pi(2j+1)/3]}{(2j+1)\sinh[\pi(2j+1)]} \approx 0.381\,V,$$

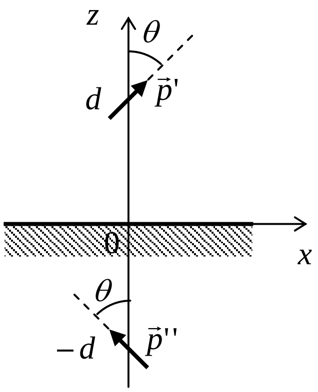
$$\Phi_B = \frac{4}{\pi} V \sum_{j=0}^{\infty} \sin\frac{\pi(2j+1)}{3} \frac{\sinh[\pi(2j+1)/3]}{(2j+1)\sinh[\pi(2j+1)]} \approx 0.119\,V.$$

Notice that though the sums are formally infinite, virtually all of their values (but a few percent) are provided by the first terms, because the sign alternation of the sin factor, as well as the exponential character of sinh, ensure a very fast convergence of the series.

Comparing the results provided by the two methods, we see that the finite difference method error, even with this rather crude mesh, is of the order of just a few percent.

Problem 6.2 (10 points). An electric dipole is located above an infinite conducting plane (see Fig. on the right). Find:

(i) the force and
(ii) the torque acting on the dipole, and
(iii) the dipole-to-plane interaction energy.



Solutions: The problem may be solved by the introduction of a dipole image, at the same distance *d* below the plane, and with the same dipole moment magnitude *p* as the original dipole, but reflected in the vertical plane perpendicular to that containing the dipole moment vector (see Fig.):[1]

Let us prove that. The net field of these two dipoles evidently satisfies the Poisson equation in the upper half-space, so that the only thing we have to prove is that this field also satisfies the boundary condition (Φ = 0) on the plane surface. Let us use Eq. (3.27) for of a system of several dipoles – in our case, of two dipoles (let us call them $\vec{p}'$ and $\vec{p}''$), with Cartesian components

$$p_x^{'} = -p_x^{''} = p\sin\theta, \quad p_y^{'} = p_y^{''} = 0, \quad p_z^{'} = p_z^{'''} = p\cos\theta,$$

located at points

$$x' = x'' = 0, \quad y' = y'' = 0, \quad z' = -z'' = d\,.$$

(Here *x* is the coordinate within the vertical plane which contains vectors $\vec{p}'$ and $\vec{p}''$, i.e. in the plane of our drawing, while axis *y* is perpendicular to the plane.) In these coordinates, Eq. (3.27) yields

$$\Phi = \frac{1}{4\pi\varepsilon_0}\left[\frac{(\vec{r}-\vec{r}')\cdot\vec{p}'}{(\vec{r}-\vec{r}')^3} + \frac{(\vec{r}-\vec{r}'')\cdot\vec{p}''}{(\vec{r}-\vec{r}'')^3}\right] = \frac{p}{4\pi\varepsilon_0}\left[\frac{(z-d)\cos\theta + x\sin\theta}{\left(x^2+y^2+(z-d)^2\right)^{3/2}} + \frac{(z+d)\cos\theta - x\sin\theta}{\left(x^2+y^2+(z+d)^2\right)^{3/2}}\right].$$

This equation shows that potential Φ vanishes at an arbitrary point of the surface (z = 0), thus proving our guess.

Now, we can use Eqs. (3.14) (or (3.15) and (3.8) of the lecture notes to calculate the potential energy of interaction force between the real and imaginary dipoles (e.g., between the dipole and the

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[1] The simplest way to understand this fact is to present the dipole in the form of two point charges (+*q*) and (-*q*), slightly displaced along the direction of the dipole moment vector, and to construct the dipole image from the mirror images of these charges in the conducting plane. However, this approach, based on only a particular implementation of a dipole, and can only be used for a *guess*, not as a *proof*.

plane). However, we should not forget that the image has been created by the dipole itself. Hence, following the reasoning of Sec. 1.3, we should multiply the RHP of Eq. (3.18) by ½:

$$U_{\text{int}} = -\frac{1}{2}\vec{p}'\cdot\vec{E}''(\vec{r}\,'). \tag{*}$$

In our coordinates, the electric field of dipole $\vec{p}''$ at point $\vec{r}\,'$ (see Eqs. (3.14) or (3.15) of the lecture notes) is

$$E_x^{''} = \frac{1}{4\pi\varepsilon_0}\frac{p\sin\theta}{(2d)^3}, \quad E_y^{''''} = 0, \quad E_z^{''} = \frac{1}{4\pi\varepsilon_0}\frac{2p\cos\theta}{(2d)^3}.$$

With that, Eq. (*) yields[2]

$$U_{\text{int}} = -\frac{1}{8\pi\varepsilon_0}\frac{p^2}{(2d)^3}\left(1+\cos^2\theta\right). \tag{**}$$

Notice that for any angle θ, the interaction energy is negative, i.e. the dipole is always attracted to a conductor. (Try to give an interpretation of this fact, presenting the dipole as a couple of point charges.)

Now we can use Eq. (**) to calculate the force and torque. As should be clear from the symmetry of this expression (namely, its independence on the horizontal position of the dipole), the force has only one component:

$$F_z = -\frac{\partial U_{\text{int}}}{\partial d} = -\frac{1}{4\pi\varepsilon_0}\frac{3p^2}{16d^4}\left(1+\cos^2\theta\right).$$

The torque vector also has only one component, but perpendicular to the plane of drawing:

$$\tau_y = -\frac{\partial U_{\text{int}}}{\partial\theta} = -\frac{1}{4\pi\varepsilon_0}\frac{p^2}{16d^3}\sin 2\theta.$$

(Notice that the same result may be obtained from Eq. (3.19).) It is interesting that the torque disappears at

(i) $\theta = 0, \pi$, and

(ii) $\theta = \pm\pi/2$,

i.e. in the positions in which the dipole moment is aligned along the field created by its image. Of these configurations, only the former two (θ = 0, π) are stable with respect to rotation, because they correspond to the interaction energy minima.

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[2] The same result follows from the equation given in the footnote on p. 4 of the lecture notes, after we it is multiplied by the same factor ½.