PHY 506

Classical Electrodynamics II

Spring 2008

Homework 5 with solutions

Problem 5.1 (20 points). For the dielectric Fabry-Perot resonator considered in Homework Problem 2.1, find the *Q*-factor due to radiation losses in the limit of strong impedance mismatch ($Z >> Z_0$), using two methods:

(i) from the frequency dependence of the power transmission coefficient,

(ii) from the energy balance.

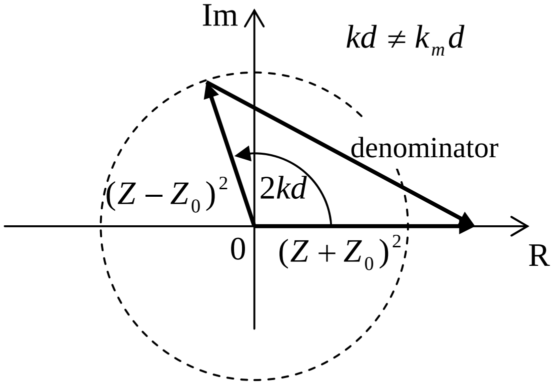
Compare the results.

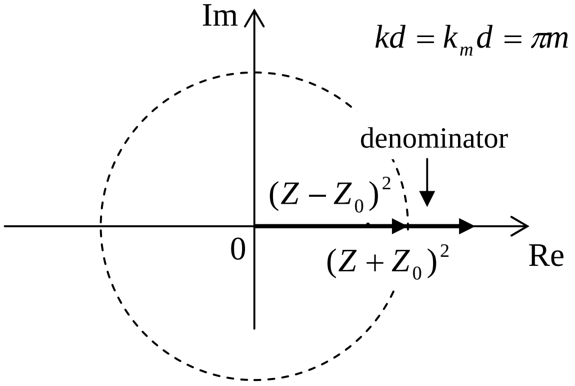
Solution:

(i) The main result of HW Problem 2.1 was the following expression for the (amplitude) transmission coefficient:

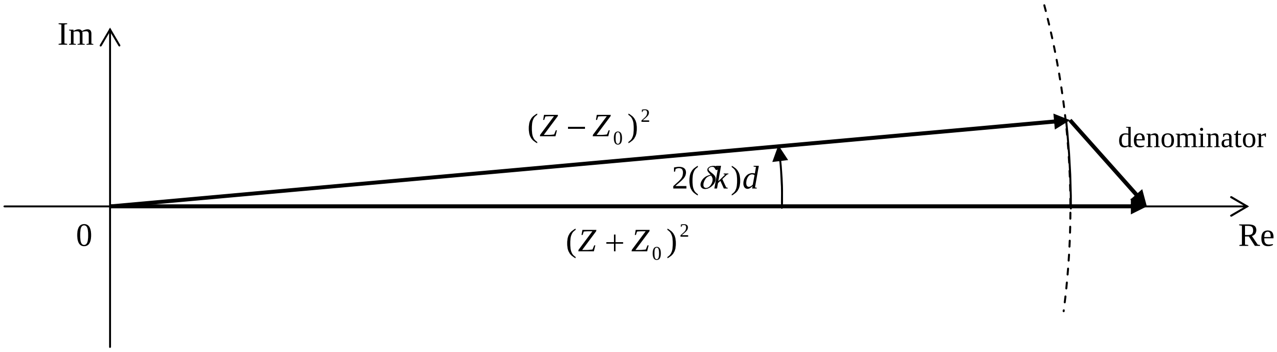
$$T = \frac{4ZZ_0}{(Z+Z_0)^2 - (Z-Z_0)^2 e^{2ikd}},$$

whose denominator can be conveniently presented by the following vector diagram,





which shows, in particular, that the resonance transmission ($T_{\max} = 1$) is reached at $k_m d = \pi m$, $m = 0, 1, 2,\ldots$ Leaving alone the special value $m = 0$ (zero frequency or zero thickness of the dielectric layer), the diagram shows that at strong impedance mismatch ($Z >> Z_0$), even a small deviation $\delta k \equiv k - k_m$ from the resonance leads to a fast growth of the denominator and hence to fast suppression of $|T|$.



As a result, in order to find the power transmission coefficient $|T|^2$ near the resonance, we may use the following approximation (see Fig. above):

$$|\text{denom}|^2 \approx \left[(Z+Z_0)^2-(Z-Z_0)^2\right]^2+(Z-Z_0)^4\left[2(\delta k)d\right]^2 \approx \left[4ZZ_0\right]^2+Z^4\left[2(\delta k)d\right]^2 .$$

The denominator increases to twice its resonance value (and hence the transmitted power drops to one half of its resonance value) when the second term in this expression becomes equal to the first one, i.e. when

$$(\delta k)\big|_{1/2}\, d = \pm\frac{2Z_0}{Z}.$$

The Q factor of the resonator is determined by the "full-width half-maximum" (FWHM) bandwidth:

$$Q=\frac{\omega_m}{\Delta\omega}=\frac{k_m d}{(\Delta k)d}=\frac{\pi m}{2|\delta k|_{1/2}\, d}=\frac{\pi m Z}{4Z_0}. \qquad (*)$$

This formula is only valid if $Q >> m$, as it is in our case $Z_0 << Z$.

(ii) The last two equations of the full system for traveling wave amplitudes T, T', R, and R' (see the models solutions for Problem 2.1) may be readily solved to give

$$T'e^{ikd}=\frac{Z_0+Z}{2Z_0}T, \quad R'e^{-ikd}=\frac{Z_0-Z}{2Z_0}T.$$

At resonance ($kd = k_m d = \pi m$), both exponential factors equal $(-1)^m$, and from Problem 2.1 we know that $T = 1$, so that at $Z >> Z_0$, $R' \approx - T'$, with

$$|T'|^2 \approx |R'|^2 \approx \left(\frac{Z}{2Z_0}\right)^2 >> 1.$$

Thus the field inside the resonator ($0 \leq z \leq d$) may be approximated with a standing wave

$$E(z)=E_\omega\left(T'e^{ikz}+R'e^{-ikz}\right)\approx -E_\omega\frac{Z}{2Z_0}\left(e^{ikz}-e^{-ikz}\right)=-iE_\omega\frac{Z}{Z_0}\sin\frac{\pi m z}{d}.$$

Notice that the wave amplitude is much higher that that (E_ω) of the incident wave. (This is due to the well known energy accumulation at resonance.). Also notable is the $\pi/2$ phase shift (presented by coefficient ($-i$) = exp{-i$\pi/2$}, also typical for the exact resonance.

Now we can use Eq. (3.74) of the lecture notes to calculate the time-average of the electric field energy in the resonator (per unit area):

$$\frac{\langle U\rangle}{A}=\frac{\varepsilon}{2}\int_0^d\left\langle E^2(z,t)\right\rangle dz=\frac{\varepsilon}{2}\frac{1}{2}|E_\omega|^2\left(\frac{Z}{Z_0}\right)^2\frac{d}{2}.$$

It is straightforward to check that the average magnetic field has an equal energy, so that the full energy of the resonator is

$$\frac{\langle E\rangle}{A} = 2\frac{\langle U\rangle}{A} = \frac{\varepsilon d}{4}\left|E_\omega\right|^2\left(\frac{Z}{Z_0}\right)^2.$$

The resonator's energy loss (per unit time) due to radiation is the sum of time-averaged Poynting vectors of outcoming waves, whose amplitudes T and R are (in our approximation) equal to each other:

$$\frac{P_{\text{loss}}}{A} = 2\langle S_T\rangle = \frac{1}{Z_0}\left|TE_\omega\right|^2,$$

so at the resonance ($T = 1$) we have

$$Q \equiv \omega_m \frac{\langle E\rangle}{\langle P_{\text{loss}}\rangle} = \omega_m \frac{\varepsilon d}{4} Z_0\left(\frac{Z}{Z_0}\right)^2 = \frac{k_m}{4Z} Z_0\left(\frac{Z}{Z_0}\right)^2 = \frac{\pi m}{4}\frac{Z}{Z_0},$$

i.e. the same result (*).

Problem 5.2 (20 points). For a rectangular resonator with dimensions $a\times b\times l$ ($b \leq a, l$), calculate the Q-factor in the basic (lowest) oscillation mode, due to the skin-effect losses in metallic walls. Evaluate the factor (and the lowest eigenfrequency) for a 23×10×23 mm^3 resonator with copper walls, at room temperature.

Solution: Selecting the coordinate origin in one of the resonator's angles, so that its interior corresponds to $0 \leq x \leq a$, $0 \leq y \leq b$, $0 \leq z < l$, the magnetic field distribution in the lowest (H_{101}) mode, discussed in class, is as follows:

$$H_x = \frac{a}{l}H_l \sin\frac{\pi x}{a}\cos\frac{\pi z}{l}, \quad H_y = 0, \quad H_z = H_l\cos\frac{\pi x}{a}\sin\frac{\pi z}{l},$$

The time-averaged energy of the EM field in the resonator may be calculated, for example, as the maximum energy of the magnetic field:

$$\langle E\rangle = \frac{\mu_0}{2}\int_0^a dx\int_0^b dy\int_0^l dz\left[H_x^2 + H_z^2\right] = \frac{\mu_0}{2}\left|H_l\right|^2 b\int_0^a dx\int_0^l dz\left[\frac{a^2}{l^2}\sin^2\frac{\pi x}{a}\cos^2\frac{\pi z}{l} + \cos^2\frac{\pi x}{a}\sin^2\frac{\pi z}{l}\right]$$
$$= \frac{\mu_0}{8}\left|H_l\right|^2 abl\left(\frac{a^2}{l^2}+1\right).$$

Now, the time-averaged power losses due to the skin-effect may be calculated virtually in the same way as in a waveguide (see, e.g., Eq. (*) in the model solutions of HW Problem 3.2):

$$\langle P_{\text{loss}}\rangle = \frac{\mu_0\omega\delta}{4}\int\limits_{\substack{\text{over all}\\ \text{the walls}}}\left|H_\tau(x,z)\right|^2 d^2r$$
$$= \frac{\mu_0\omega\delta}{4}\left\{2b\int_0^a\left|H_x(x,0)\right|^2 dx + 2b\int_0^l\left|H_z(0,z)\right|^2 dz + 2\int_0^a dx\int_0^l dz\left[\left|H_x(x,z)\right|^2 + \left|H_z(x,z)\right|^2\right]\right\}$$
$$= \frac{\mu_0\omega\delta}{4}\left|H_l\right|^2 l\left\{b\left(\frac{a^3}{l^3}+1\right) + \frac{a}{2}\left(\frac{a^2}{l^2}+1\right)\right\}.$$

As a result, for the Q-factor we get

$$Q \equiv \omega \frac{\langle E \rangle}{\langle P_{\text{loss}} \rangle} = \frac{1}{2\delta} \left\{ ab\left(\frac{a^2}{l^2} + 1 \right) \right\} / \left\{ b\left(\frac{a^3}{l^3} + 1 \right) + \frac{a}{2}\left(\frac{a^2}{l^2} + 1 \right) \right\}.$$

As expected, Q scales as the ratio of some effective size of the resonator (of the order of wavelength λ), to skin depth δ.

The resonance frequency may be found from the wave vector $k \equiv \omega/c$ which should satisfy the resonance condition:

$$k^2 = \left(\frac{\pi}{a} \right)^2 + \left(\frac{\pi}{l} \right)^2,$$

so that

$$f = \frac{\omega}{2\pi} = \frac{ck}{2\pi} = \frac{c}{2}\sqrt{\frac{1}{a^2} + \frac{1}{l^2}}.$$

For the resonator dimensions given in the assignment, the last equation yields $f \approx 9.22\times10^9$ Hz ≈ 10 GHz, so that the skin depth for copper is close to that ($\delta \approx 6.5\times10^{-7}$ m) calculated in Homework 3, and $Q \approx 8.2\times10^3$. For microwave, room-temperature, metallic resonators, this is almost as high quality as you can get. (The numbers about 10^3 are more typical.) Notice that the condition $Q >> 1$ is well satisfied, so that our approximate method of calculation is indeed legitimate.