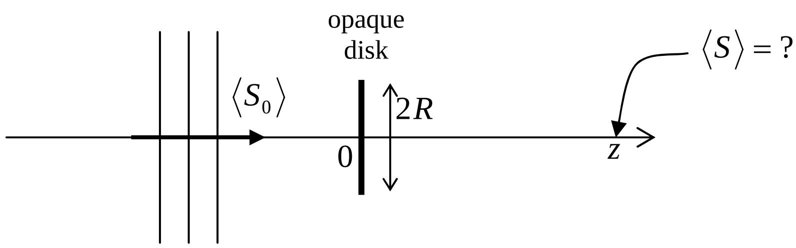
PHY 506 **Classical Electrodynamics II** Spring 2008

Homework 7 with solutions

Problem 7.1 (20 points). A plane monochromatic wave is normally incident on an opaque disk of radius $R >> \lambda$. Use the Huygens principle to calculate the wave intensity at distance $z >> R$ behind the disk center (see Fig. below).



Solution: According to the Huygens principle, the complex amplitude in the observation point is

$$f = f_0 \frac{k}{2\pi i} \int_R^\infty \rho' d\rho' \int_0^{2\pi} d\varphi' \frac{\exp\left\{ik\sqrt{\rho'^2 + z^2}\right\}}{\sqrt{\rho'^2 + z^2}}.$$

Using the condition, $z >> R >> \lambda$, this expression may be simplified, just as we did in class:

$$f \approx f_0 \frac{k}{iz} e^{ikz} \int_R^\infty \exp\left\{i \frac{k\rho'^2}{2z}\right\} \rho' d\rho' = f_0 \frac{k}{iz} e^{ikz} \frac{z}{k} \int_{kR^2/2z}^\infty \exp\{i\xi\} d\xi = -f_0 e^{ikz} \exp\{i\xi\}\Big|_{kR^2/2z}^{\infty}.$$

Since $\exp\{i\xi\} = \cos\xi + i \sin\xi$ is an oscillating rather than a decaying function, the result of the upper-limit substitution may not be apparent. However, since this is an analytical function, we can always add to ξ a small imaginary part $i\zeta$. As a result, the upper-limit substitution disappears, and now we can make the transition $\zeta \to 0$. As a result, we get

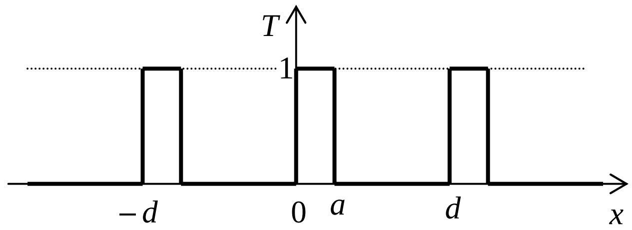
$$f(x, z) = f_0 e^{ikz} \exp\left\{i \frac{kR^2}{2z}\right\},$$

leading to a paradoxical answer

$$\langle S \rangle = \langle S_0 \rangle.$$

Historically, this result was first obtained in 1818 by Poisson who used it as an argument against the theory of diffraction which had been developed by Fresnel (and against the wave theory of light as the whole). The following experiments have shown, however, that the diffraction pattern on a disk with $R << z$ does indeed feature a light spot in the center.

Problem 7.2 (20 points). Within the Fraunhofer approximation, analyze the pattern produced by a 1D diffraction lattice with the periodic transparency profile shown below, upon the normal incidence of a plane wave.



Solution: As discussed in Sec. 9.7 of the lecture notes, within the Fraunhofer approximation (valid when distance z between the lattice and the observer is much larger than d^2/λ), the diffraction pattern $f(\vec{r})$ is just the Fourier transform of the wave amplitude $f(\vec{r}\,')$ at the lattice, i. e. of its transparency $T(\vec{r}\,') = f(\vec{r}\,')/f_0$. In our 1D case we have

$$f(\vec{r}) \to f(x) = \text{const} \times \int_{-\omega}^{+\infty} T(x')\exp\{ik_t x'\}dx' = \text{const} \times \int_{-\omega}^{+\infty} T(x')\exp\left\{i\frac{kx}{z}x'\right\}dx'.$$

Since in our case the transparency is periodic, $T(x' + d) = T(x')$, the intensity is non-vanishing only for a discrete set of equidistant values

$$k_n = \frac{2\pi}{d}n, \qquad x_n = \frac{z}{k}k_n = \frac{2\pi z}{kd}n, \quad n = 0, \pm 1, \pm 2, ...$$

which produce on a screen as a set of thin "interference stripes" parallel to those on the diffraction lattice. For these values,

$$f_n = \text{const} \times \int_0^d T(x')\exp\left\{2\pi i\frac{x'}{d}n\right\}dx' \propto \int_0^a \exp\left\{2\pi i\frac{x'}{d}n\right\}dx'$$
$$= \frac{d}{2\pi i n}\left[\exp\left\{2\pi i\frac{x'}{d}n\right\}\right]_{x'=0}^{x'=a} = \frac{d}{2\pi i n}\left[\exp\left\{2\pi i\frac{a}{d}n\right\} - 1\right].$$

For analysis, its is convenient to present this complex amplitude as a product of a real function by a phase factor which does not affect the wave intensity:

$$f_n \propto \frac{d}{\pi n}\sin\frac{\pi a n}{d}\exp\left\{i\pi\frac{a}{d}n\right\} \propto \text{sinc}\frac{\pi a n}{d}\exp\left\{i\pi\frac{a}{d}n\right\}.$$

Thus the stripe intensity is modulated by the "envelope function"

$$|f_n|^2 \propto \text{sinc}^2\frac{\pi a n}{d}.$$

(For a plot, see Sec. 9.4 of the lecture notes.) The function decreases rapidly (as n^{-2}) as soon as its argument' magnitude exceeds $\sim\pi/2$, so that the number of stripes of comparable intensity $\Delta n \sim d/a + 1$, the unity counting the central stripe ($n = 0$).