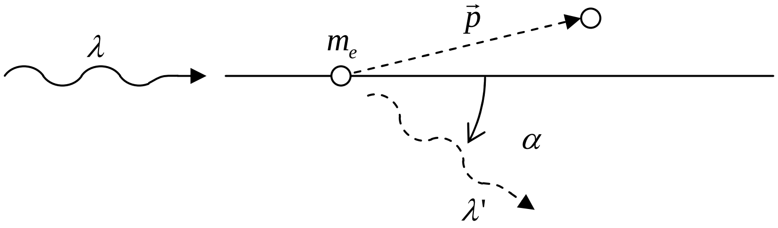
PHY 506 **Classical Electrodynamics II** Spring 2008

Homework 8 with solutions

Problem 8.1 (10 points). Photon with wavelength λ is scattered by an electron, initially at rest. Considering the photon as an ultrarelativistic particle (with the rest mass $m = 0$), find wavelength λ' of the scattered phonon as a function of the scattering angle α (see Fig. below).



Solution: This the famous *Compton scattering* problem. It may be solved, for example, by writing the laws of conservation of the total energy and momentum, in the lab reference frame:

$$\hbar\omega + m_e c^2 = \hbar\omega' + \sqrt{\left(m_e c^2\right)^2 + p^2 c^2}\ ,$$

$$\hbar\vec{k} = \hbar\vec{k}' + \vec{p},$$

where $\vec{p}$ is the momentum of the electron after the collision (see Fig. above), and $k = \omega/c = 2\pi/\lambda$. From the last equation, we can write

$$p^2 = \hbar^2(\vec{k} - \vec{k}')^2 = \hbar^2\left(k^2 + k'^2 - 2kk'\cos\alpha\right).$$

(Notice that this is a very useful way to avoid the introduction of the direction of vector $\vec{p}$.) Plugging this expression into the first equation, and solving for λ', we get

$$\lambda' = \lambda + \frac{2\pi\hbar}{m_e c}\left(1 - \cos\alpha\right) \geq \lambda\,.$$

The only parameter participating in this result,

$$\frac{2\pi\hbar}{m_e c} = \frac{h}{m_e c} \approx 2.43\times 10^{-12}\,\mathrm{m}\,,$$

is frequently called the "Compton wavelength", though this name is somewhat misleading. (No particle in the problem has this wavelength.)

Problem 8.2 (10 points). An atom with the rest mass m, initially at rest and internally excited by energy $\Delta\mathcal{E}$, transitions to its ground state, emitting a photon. Find the photon's frequency, taking into account the relativistic recoil of the atom.

Solution: Combining the energy conservation law,

$$mc^2 + \Delta\mathcal{E} = \hbar\omega + \sqrt{(mc^2)^2 + p^2c^2}\ ,$$

and the momentum conservation law

$$0 = \frac{\hbar\omega}{c} - p$$

(both in the lab frame, which is also the center-of-mass frame in this problem), we readily get

$$\omega = \frac{\Delta\mathcal{E}}{\hbar} \times \frac{1 + \Delta\mathcal{E}/2mc^2}{1 + \Delta\mathcal{E}/mc^2} \approx \frac{\Delta\mathcal{E}}{\hbar} \times \left(1 - \frac{\Delta\mathcal{E}}{2mc^2}\right) < \frac{\Delta\mathcal{E}}{\hbar}.$$

For the real atoms ($m \sim 10^{-26}$ kg, $\Delta\mathcal{E} \sim 1$ eV), the recoil correction is small ($\sim 10^{-10}$), but still may be noticeable for quantum transitions with small internal linewidth. This correction is dramatically reduced in the Mössbauer effect in which the role of m is a played by the mass of the whole crystal lattice.

Problem 8.3 (10 points). Find the threshold energy of a γ-photon for the reaction

$$\gamma + p \to p + \pi^0,$$

if the proton p is initially at rest.

Solution: Similarly to our analysis of $p + p \to p + p + p + \bar{p}$ reaction in class, we can present the conservation of the net 4-momentum as

$$(\gamma + p)_\alpha(\gamma + p)^\alpha = (p + \pi^0)_\alpha(p + \pi^0)^\alpha.$$

After opening the parentheses, we can take the Lorentz-invariant scalar product $p_\alpha p^\alpha$ in the LHP equal to $m_p{}^2c^2$, and the similar product the γ-photon for zero, because it does not have rest mass. Similarly, the RHP equals to $(m_p + m_\pi)^2c^2$, because at the reaction threshold its products have zero velocity in the c.o.m. frame. From this, we get

$$2\gamma_\alpha p^\alpha = \left[(m_p + m_\pi)^2 - m_p^2\right]c^2.$$

In the lab system, in which the initial proton rests, its 4-vector p^α has only one nonvanishing component $p_0 = m_pc$, while the corresponding component γ_0 for the photon is just $\mathcal{E}_\gamma/c$. As a result, we get

$$\mathcal{E}_\gamma = \frac{1}{2m_p}\left[(m_p + m_\pi)^2 - m_p^2\right]c^2 = m_\pi c^2\left(1 + \frac{m_\pi}{2m_p}\right).$$

The second term in the parentheses represents the energy price we have to pay for not having the center of mass of the initial particles at rest. For protons $m_pc^2 \approx 938$ MeV, while for neutral pions $m_\pi c^2 \approx 135$ MeV, so that the correction is close to 7% (quite noticeable!), and finally we get $\mathcal{E}_\gamma \approx 145$ GeV.