PHY 506 **Classical Electrodynamics II** Spring 2008

Homework 9 with solutions

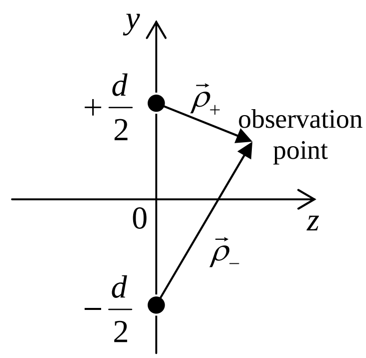
Problem 9.1 (20 points). Each of two very thin, long, parallel beams of electrons of the same velocity *u* carries electric charge of density λ per unit length (as observed in the coordinate frame moving with electrons).

(i) Calculate the distribution of the electric and magnetic fields in the system (outside the beams), as measured in the lab frame.

(ii) Calculate the interaction force between the beams (per particle) and the resulting acceleration, both in the lab frame, and in the system moving with the electrons. Compare the results and give a brief discussion of the comparison.

Solutions:

(i) In the reference frame moving with electrons, they are static, so there is no magnetic field: $\vec{B}'=0$. The electric field observed in that frame may presented as a sum of two fields (each created by one beam),



$$\vec{E}' = \vec{E}'_+ + \vec{E}'_-,$$

and each of these components may be readily found, say, from the Gauss theorem applied to a round cylinder of radius $\rho_\pm$ (see Fig. on the right), with the corresponding beam serving as an axis:

$$\vec{E}'_\pm = \frac{\lambda \vec{\rho}_\pm}{2\pi\varepsilon_0 \rho_\pm^2},$$

or, with the coordinate choice shown in Fig.,

$$E'_{\pm x} = 0, \quad E'_{\pm y} = \frac{\lambda(y \mp d/2)}{2\pi\varepsilon_0\left[(y \mp d/2)^2 + z^2\right]}, \quad E'_{\pm z} = \frac{\lambda z}{2\pi\varepsilon_0\left[(y \mp d/2)^2 + z^2\right]}.$$

Now using the Lorentz transform for the fields, derived in class, in the lab system we also can write

$$\vec{E} = \vec{E}_+ + \vec{E}_-,$$

with[1]

$$E_{\pm x} = 0, \quad E_{\pm y} = \gamma E'_{\pm y} = \gamma \frac{\lambda(y \mp d/2)}{2\pi\varepsilon_0\left[(y \mp d/2)^2 + z^2\right]}, \quad E_{\pm z} = \gamma E'_{\pm z} = \gamma \frac{\lambda z}{2\pi\varepsilon_0\left[(y \mp d/2)^2 + z^2\right]},$$

$$B_{\pm x} = 0, \quad B_{\pm y} = -\frac{\gamma u}{c^2} E'_{\pm z} = -\frac{\gamma u}{c^2} \frac{\lambda z}{2\pi\varepsilon_0\left[(y \mp d/2)^2 + z^2\right]}, \quad B_{\pm z} = \frac{\gamma u}{c^2} E'_{\pm y} = \frac{\gamma u}{c^2} \frac{\lambda(y - d/2)}{2\pi\varepsilon_0\left[(y \mp d/2)^2 + z^2\right]},$$

$$\gamma \equiv \frac{1}{\sqrt{1 - u^2/c^2}}.$$

[1] The Lorentz transform does not change length perpendicular to the relative velocity, so that *y' = y*, *z' = z*.

(ii) The Lorenz force acting on one beam (say, the one located at $y = +d/2$) comes only from the fields created by the other beam (located at $y' = - d/2$). As a result, in the frame moving with the particles

$$F_y' = qE_{-y}'\Big|_{y=d/2,\, z=0} = \frac{q\lambda}{2\pi\varepsilon_0 d}, \quad F_z' = 0\,,$$

while in the lab frame the magnetic field contributes to the force as well:

$$\vec{F} = q(\vec{E}_- + \vec{u}\times\vec{B}_-)_{y=d/2,\, z=0},$$

and we get

$$F_y = q(E_{-y} - uB_{-z}) = q\gamma\frac{\lambda}{2\pi\varepsilon_0 d}\left(1-\frac{u^2}{c^2}\right) = \frac{1}{\gamma}\frac{q\lambda}{2\pi\varepsilon_0 d}, \quad F_z = q(E_{-z} + uB_{-y}) = 0\,.$$

The resulting vertical acceleration in the moving frame (where $M = m$) is

$$a_y' = \frac{F_y'}{m} = \frac{q\lambda}{2\pi\varepsilon_0 dm},$$

while in the lab frame, where $M = \gamma m$, it is

$$a_y = \frac{F_y}{M} = \frac{1}{\gamma^2}\frac{q\lambda}{2\pi\varepsilon_0 dm} = \frac{a_y'}{\gamma^2}\,.$$

These results for F_y and a_y are dramatically different, but they are actually consistent, because time runs differently in the two frames. For example, if the acceleration produces a small beam shift $\Delta y << d$, with the vertical velocity still much below c, we can write

$$\Delta y' = \frac{a_y'}{2}(\Delta t')^2 = \frac{a_y'}{2}(\Delta\tau)^2, \qquad \Delta y = \frac{a_y}{2}(\Delta t)^2 = \frac{a_y'}{2\gamma^2}(\Delta t)^2.$$

Since the "proper" time interval $\Delta t' = \Delta\tau$ and the lab frame interval Δt are related as $\Delta t = \gamma\Delta\tau$ ("time dilation"), we get that $\Delta y' = \Delta y$, as it should be.

Problem 9.2 (20 points). Find the trajectory of a relativistic particle in a uniform electrostatic field $\vec{E}$ for the case of arbitrary initial velocity $\vec{u}(0)$.

Hint: You may like to explore ways of integration the equation of motion, different from the one used in class for the case $\vec{u}(0) \perp \vec{E}$.

Solutions: (i) An elegant alternative way to solve this problem is to integrate the 4-vector equation derived in class,

$$\frac{dp^\alpha}{d\tau} = qF^{\alpha\beta}U_\beta,$$

directly, considering the proper time τ of the particle as an argument. For the nonvanishing components of 4-velocity[2] we get equations

$$\frac{d(\gamma c)}{d(\Gamma\tau)} = \gamma u_z, \quad \frac{d(\gamma u_x)}{d(\Gamma\tau)} = 0, \quad \frac{d(\gamma u_z)}{d(\Gamma\tau)} = \gamma c,$$

where $\Gamma \equiv qE/cm$ is a constant parameter with the reciprocal time (s^{-1}) dimensionality. The middle equation is elementary, and yields

$$\gamma u_x = \text{const} = \frac{cu_x(0)}{\sqrt{c^2 - u_x^2(0) - u_z^2(0)}} \equiv C.$$

The remaining two equations may be combined (by the additional differentiation of any of them over τ and substitution of the remaining equation) to give similar second-order differential equations

$$\frac{d^2}{d\tau^2}(\gamma c) = \Gamma^2(\gamma c), \quad \frac{d^2}{d\tau^2}(\gamma u_z) = \Gamma^2(\gamma u_z),$$

with similar solutions

$$\gamma c = A\cosh\Gamma(\tau - \tau_0), \quad \gamma u_z = A\sinh\Gamma(\tau - \tau_0). \qquad (*)$$

Constants A and τ_0 may be found from the initial conditions:

$$A = c\sqrt{\frac{c^2 - u_z^2(0)}{c^2 - u_x^2(0) - u_z^2(0)}} = c\frac{\gamma(0)}{\gamma_z(0)}, \qquad \gamma_z(0) \equiv \frac{1}{\sqrt{1 - u_z^2(0)/c^2}}, \quad \tau_0 = -\frac{1}{\Gamma}\operatorname{arctanh}\frac{u_z(0)}{c}.$$

Physically, τ_0 is the proper time at which the particle reaches the lowest value of z.

Now we can find the tangent to trajectory as

$$\frac{dz}{dx} = \frac{u_z}{u_x} = \frac{A}{C}\sinh\Gamma(\tau - \tau_0),$$

but in order to integrate this equation we still need to express its RHP as a function of either x or z. For example, we can find $z(\tau)$, integrating the second of Eqs. (*):

$$\gamma u_z \equiv \gamma\frac{dz}{dt} = \frac{dz}{d\tau} = A\sinh\Gamma(\tau - \tau_0),$$

$$z = A\int_0^\tau \sinh\Gamma(\tau - \tau_0)d\tau = \frac{A}{\Gamma}\left[\cosh\Gamma(\tau - \tau_0) - \cosh\Gamma\tau_0\right].$$

From here,

$$\cosh\Gamma(\tau - \tau_0) = \frac{\Gamma z}{A} + \cosh\Gamma\tau_0 = \frac{\Gamma z}{A} + \frac{\gamma(0)c}{A} = \frac{\Gamma z}{A} + \gamma_z(0),$$

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[2] We are using the same coordinate system choice as in class, with axis z along the electric field, and axis x in the plane of motion, so that $u_y = 0$.

$$\sinh\Gamma(\tau-\tau_0)=\sqrt{\left(\frac{\Gamma z}{A}+\gamma_z(0)\right)^2-1}\,,$$

and now we can finally find the trajectory:

$$\frac{dz}{dx}=\frac{A}{C}\sinh\Gamma(\tau-\tau_0)=\frac{1}{C}\sqrt{\left(\Gamma z+A\gamma_z(0)\right)^2+A^2}\,,$$

$$x=C\int_0^z\frac{dz}{\sqrt{\left(\Gamma z+A\gamma_z(0)\right)^2-A^2}}=\frac{C}{\Gamma}\left[\operatorname{arccosh}\left(\frac{\Gamma z}{A}+\gamma_z(0)\right)-\operatorname{arccosh}\gamma_z(0)\right],$$

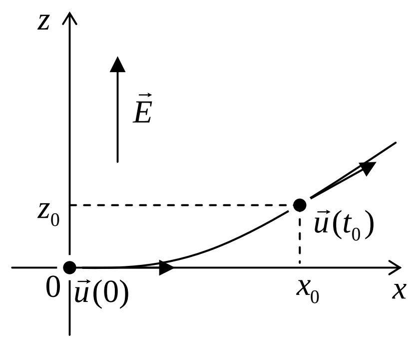
which is a natural generalization of the result derived in class for the particular case $u_z(0) = 0$ and hence $\gamma_z(0) = 1$, $A = c\gamma(0)$, $A/\Gamma = \gamma(0)mc^2/qE = \mathcal{E}(0)/qE$, $C/\Gamma = p_0c/qE$, so that

$$x=\frac{p_0c}{qE}\left[\operatorname{arccosh}\left(\frac{qEz}{\mathcal{E}_0}+1\right)\right].$$

(ii) Another way to solve this problem is to notice that the formula calculated in class,

$$z=\frac{\mathcal{E}_0}{qE}\left(\cosh\frac{qEx}{cp_0}-1\right),$$

for the case $\vec{u}(0)\perp\vec{E}$, may be used in the general case as well, if we shift the origins of x, z, and t (see Fig. on the right),



$$z=z_0+\frac{\mathcal{E}(0)}{qE}\left(\cosh\frac{qE(x-x_0)}{cp(0)}-1\right)$$

and the "only" thing we should express parameters $\mathcal{E}(0)$, $p(0)$, x_0, and z_0 via components $u_x(0)$, $u_z(0)$of the new initial velocity, which in this notation becomes $\vec{u}(t_0)$. However, the recalculation (leading of course to the same result as above) is actually not easier then the first way.