PHY 506 **Classical Electrodynamics II** Spring 2008

Homework 11 with solution

Problem 11.1 (20 points). Find the law of motion of a relativistic particle in parallel, static electric and magnetic fields.

Hint: Try use the proper time of the particle. (See, e.g., the model solution of Problem 9.2.)

Solution: For an arbitrary direction of the initial momentum $\vec{p}(0)$ of the particle, we can always select the coordinate frame so that

$$\vec{E} = \{0,0,E\}, \quad \vec{B} = \{0,0,B\}, \quad \vec{p}(0) = \{p_x(0),0,p_z(0)\}.$$

Then the 4-vector equation of motion of the particle

$$\frac{dU^\alpha}{d\tau} = \frac{q}{m} F^{\alpha\beta} U_\beta$$

may be separated into components as

$$\frac{d(\gamma c)}{d\tau} = \Gamma(\gamma u_z), \quad \frac{d(\gamma u_x)}{d\tau} = -\omega(\gamma u_y), \quad \frac{d(\gamma u_y)}{d\tau} = \omega(\gamma u_x), \quad \frac{d(\gamma u_z)}{d\tau} = \Gamma(\gamma c),$$

where $\Gamma \equiv qE/cm$, and $\omega \equiv qB/m$. (Notice that the last expression is *not* the usual relativistic cyclotron frequency $\omega_c = qB/\gamma m$: its denominator does not include factor γ, as it would if we dealt with the lab time t).

Now we see that the first and the last equations of the system are not affected by the magnetic field and are exactly the same as those solved in Problem 9.2. On the other hand, the equations u_x and u_y independent on the electric field, and are exactly the same as for the nonrelativistic cyclotron motion.[1] They can be readily solved (by the usual trick of differentiation of one of them over τ, and then plugging the counterpart equation into the result) to give:

$$\gamma u_x = \frac{p_x(0)}{m}\cos\omega\tau, \quad \gamma u_y = \frac{p_x(0)}{m}\sin\omega\tau.$$

An elementary integration of these equations over τ, with the initial point taken for $x = y = 0$, yields

$$x = R\sin\omega\tau, \quad y = R(\cos\omega\tau - 1),$$

where the cyclotron orbit radius $R \equiv p_x(0)/qB$ is not affected by the initial momentum of the particle in z direction.

[1] Notice that this separation of the orbital motion from the motion in the accelerating electric field is only possible in proper time! In the lab time, the particle becomes heavier as it is accelerated, and its cyclotron frequency decreases: $\omega_c(t) = \omega/\gamma(t)$.