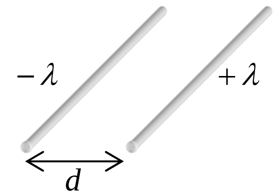
PHY 505 **Classical Electrodynamics I** Fall 2007

Midterm Exam 1 with Solutions

Problem M.1 (125 points.) Two thin, straight parallel wires, separated by distance d, are charged by equal and opposite uniformly distributed charges with linear density λ - see Fig. on the right. Find the electrostatic force (per unit length) of interaction between the wires. Compare the result with the Coulomb Law for the force between the point charges, and interpret their difference.



Solution: Let us apply the Gauss Law to a cylinder of unit length, coaxial with one of the wires (say, the one carrying charge $+\lambda$), and radius equal to d. The electric field (created by that wire alone!) is evidently perpendicular to the cylinder surface, and its magnitude E is the same in all points of the surface. As a result, the Gauss Law yields

$$2\pi d E = \frac{\lambda}{\varepsilon_0},$$

so that the attractive force between the wires

$$\frac{F}{L} = \lambda E = \frac{\lambda^2}{2\pi\varepsilon_0 d}.$$

This result shows that, in contrast with the Coulomb force between the point charges, the force between the wires falls slower (as $1/d$ rather than as $1/d^2$) as the distance between them is increased. This may be interpreted as a result of an elementary charge $dQ = \lambda dx$ of one wire "seeing" a larger length of another wire as d grows. (We have seen a similar, but even stronger "horizon increase" effect for a field created by a plane – see Fig. 1.3 and its discussion.)

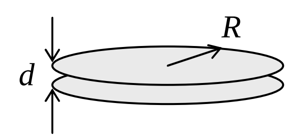
Problem M.2 (125 points.) Using the results for a single thin round disk, obtained in class, consider a system of two such disks at a small distance $d << R$ from each other - see Fig. on the right. In particular, calculate:

(i) the reciprocal capacitance matrix of the system,

(ii) the mutual capacitance between the disks,

(iii) the partial capacitance, and

(iv) the effective capacitance of one disk,



(all in the first non-vanishing approximations in $d/R << 1$). Compare the results (ii)-(iv) and interpret their similarities and differences.

Solution:

An arbitrary set of disk charges, Q_1 and Q_2, may be presented as a sum of a symmetric and antisymmetric distributions:

$$Q_{1,2} = Q_s \pm Q_a,$$

with

$$Q_s = \frac{Q_1 + Q_2}{2}, \quad Q_a = \frac{Q_1 - Q_2}{2}. \qquad (*)$$

The linear superposition principle tells us that the electrostatic potential distribution (and hence the potential of each disk) may be found as a sum of the components created by the corresponding charge distributions:

$$\Phi_{1,2} = \Phi_s \pm \Phi_a. \qquad (**)$$

Let us first consider the symmetric distribution: $Q_1 = Q_2 = Q_s$. In this case the small space between the disks has almost no electric field and hence does not contribute to the system capacitance. The field outside has been calculated in class, and gives total capacitance

$$C \equiv \frac{Q_1 + Q_2}{\Phi_s} = \frac{2Q_s}{\Phi_s} = 8\varepsilon_0 R,$$

so that

$$\Phi_s = \frac{Q_s}{4\varepsilon_0 R}.$$

On the opposite, the antisymmetric distribution of charge, $Q_1 = - Q_2 = Q_a$, does not create electric field outside the system (because the net charge of the disks is zero), while the field between them is virtually the same as in the plane capacitor, with the well-known mutual capacitance (also calculated in class)

$$C_m = \frac{\varepsilon_0 A}{d} = \frac{\pi \varepsilon_0 R^2}{d}. \qquad (***)$$

It is important that this capacitance, by definition, relates the charge *one* disk (say, $Q_1 = Q_a$) to voltage V, i.e. the difference of potentials between the conductors, $V = \Phi_1 - \Phi_2$, which, in this antisymmetric case ($\Phi_1 = -\Phi_2 = \Phi_a$) equals to $2\Phi_a$. As a result,

$$\Phi_a = \frac{V}{2} = \frac{Q_a}{2C_m} = \frac{Q_a d}{2\pi\varepsilon_0 R^2}.$$

Plugging the results for Φ_s and Φ_a into Eq. (**), and using Eq. (*) for Q_s and Q_a, we finally get

$$\Phi_1 = \frac{1}{4\varepsilon_0 R} Q_s + \frac{d}{2\pi\varepsilon_0 R^2} Q_a = \left(\frac{1}{8\varepsilon_0 R} + \frac{d}{4\pi\varepsilon_0 R^2}\right) Q_1 + \left(\frac{1}{8\varepsilon_0 R} - \frac{d}{4\pi\varepsilon_0 R^2}\right) Q_2,$$

$$\Phi_2 = \frac{1}{4\varepsilon_0 R} Q_s - \frac{d}{2\pi\varepsilon_0 R^2} Q_a = \left(\frac{1}{8\varepsilon_0 R} - \frac{d}{4\pi\varepsilon_0 R^2}\right) Q_1 + \left(\frac{1}{8\varepsilon_0 R} + \frac{d}{4\pi\varepsilon_0 R^2}\right) Q_2.$$

This means that the reciprocal capacitance matrix components equal

$$p_{11} = p_{22} = \frac{1}{8\varepsilon_0 R} + \frac{d}{4\pi\varepsilon_0 R^2}, \quad p_{12} = p_{21} = \frac{1}{8\varepsilon_0 R} - \frac{d}{4\pi\varepsilon_0 R^2}.$$

The mutual capacitance C_m has been already found above – see Eq. (***), but as a sanity check we may verify that the result is the same if we put the above expressions for p_{jk} into Eq. (2.24) of the lecture notes:

$$C_\mathrm{m} \equiv \frac{1}{p_{11} + p_{22} - p_{12} - p_{21}}.$$

Next, the partial capacitance of one disk is

$$C_1 = C_2 = \frac{1}{p_{11}} = \frac{1}{p_{22}} \approx 8\varepsilon_0 R .$$

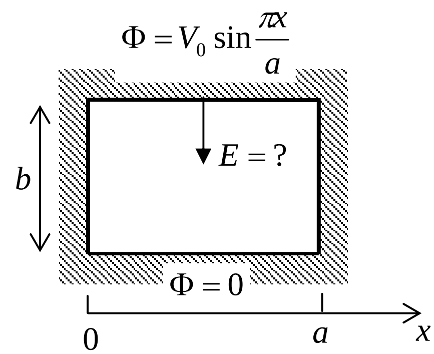
Since $d << R$, this capacitance is much lower than C_m. Finally, the effective capacitance of one disk (with another one grounded) is

$$C_1^\mathrm{ef} \equiv \left(p_{11} - \frac{p_{12}\, p_{21}}{p_{22}} \right)^{-1} \approx \frac{\pi\varepsilon_0 R^2}{d}.$$

In our approximation, this is just the mutual capacitance C_m between the disks. In the system of two close conductors, this is natural – see Eq. (2.34) of the lecture notes, and its discussion.

Notice that if only the principal terms were kept in the expressions for p_{jk}, the results for the mutual capacitance and partial capacitances would be wrong (infinite).

Problem M.3 (150 points). Apply the variable separation method to find the electric field in the middle of the top lid of a cylindrical box with the cross-section shown in Fig. on the right. The electrostatic potential of the bottom lid and side walls of the box is zero, while that of the top lid follows the sinusoidal function specified in the figure.



Solution: The separation of variables is carried out just as for the 3D box in class, giving

$$\Phi(x, y) = \sum_{n=0}^{\infty} C_n \sin\frac{\pi n x}{a} \sinh\frac{\pi n y}{a}.$$

Comparing this solution, taken on the top lid ($y = b$), with the fixed potential of the lid, we see that only one coefficient C_n survives:

$$V_0 = C_1 \sinh\frac{\pi b}{a},$$

so that the final distribution of the potential inside the box is

$$\Phi(x, y) = V_0 \sin\frac{\pi x}{a} \frac{\sinh\dfrac{\pi y}{a}}{\sinh\dfrac{\pi b}{a}}.$$

From this solution, electric field in the middle of the top lid is

$$E(\frac{a}{2},b)=\left|-\frac{\partial\Phi}{\partial y}\right|_{\substack{x=a/2,\\ y=b}}=V_0\frac{\pi}{a}\operatorname{cotanh}\frac{\pi b}{a}.$$

If the box is tall and/or narrow ($b >> a$), then cotanh($\pi b/a$) ≈ 1, and the electric field does not depend on b: $E \approx \pi V_0/a$, while in the opposite limit it is independent of a: $E \approx V_0/b$. (In the latter case the system is just the usual plane capacitor.)