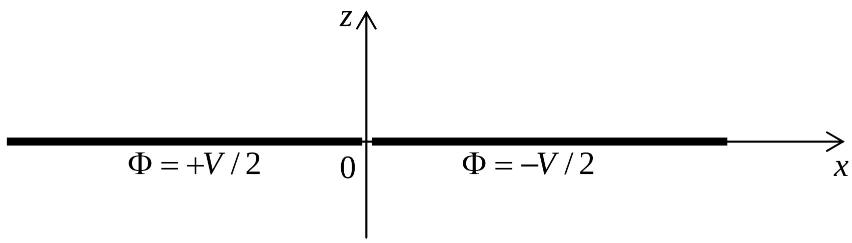
PHY 505 **Classical Electrodynamics I** Fall 2007

Midterm Exam 2 with Solutions

Problem M.4 (to be graded of 200 points). A conducting plane located at $z = 0$ is separated into two parts with a very narrow, straight cut along axis y, and voltage V is fixed between the resulting half-planes, as shown in Fig. below. Find the distribution of the electrostatic potential in all the space, and the electric field on the symmetry plane ($x = 0$).



Solution: For our cylindrical geometry ($\partial/\partial y = 0$), and in the absence of free charges, the basic formula of the Green's function theory is reduced to

$$\Phi(\vec{\rho}) = \frac{1}{4\pi}\sum_k \Phi_k \oint_{L_k} \frac{\partial G(\vec{\rho}, \vec{\rho}\,')}{\partial n'} dl',$$

where, for our choice of coordinates, $\vec{\rho} = \{x, z\}$, while $G(\vec{\rho}, \vec{\rho}\,')$ is the 2D Green's function, and the integration should be extended along all boundaries of conductors' cross-sections (numbered with index k).[1] For a semi-space limited by a conducting plane, the Green's function and its normal derivative have been calculated in Homework Problem 5.4,

$$G(\vec{\rho}, \vec{\rho}\,') = -2\ln\left|\vec{\rho} - \vec{\rho}\,'\right| + 2\ln\left|\vec{\rho} - \vec{\rho}\,''\right| = -\ln\left[(x - x')^2 + (z - z')^2\right] + \ln\left[(x - x')^2 + (z + z')^2\right], \quad (*)$$

$$\frac{\partial G}{\partial z'}\Big|_{z'=0} = \frac{4z}{(x - x')^2 + z^2},$$

so that for our current problem we get

$$\Phi = \frac{(+V/2)}{4\pi}\int_{-\infty}^{0} \frac{4y}{(x - x')^2 + z^2} dx' + \frac{(-V/2)}{4\pi}\int_{0}^{+\infty} \frac{4y}{(x - x')^2 + z^2} dx' = -\frac{V}{\pi}\arctan\frac{x}{|z|}.$$

On the symmetry plane ($x = 0$), $\Phi = 0$, so that the electric field does not a vertical component. Its only, horizontal component

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[1] The problem could be also solved using the 3D Green's function,

$$G(\vec{r}, \vec{r}\,') = \frac{1}{\sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2}} - \frac{1}{\sqrt{(x - x')^2 + (y - y')^2 + (z + z')^2}},$$

but this would cost us one more integration (along the y axis) which would bring us right back to Eq. (*).

$$E_x = -\frac{\partial \Phi}{\partial x}\Big|_{x=0} = \frac{V}{\pi |z|}.$$

Problem M.5 (100 points). In two separate experiments, a thin, plane sheet of a dielectric material with ε_r = const is placed into a uniform external electric field $\vec{E}_0$:

(i) with the sheet surface parallel to the electric field, and

(ii) the surface perpendicular to the field.

For each case, find the electric field $\vec{E}$, the electric displacement $\vec{D}$, and the polarization vector $\vec{P}$ inside the dielectric.

Solution: The same reasoning which was applied in class to vacuum slits in dielectrics, gives the following results:

(i) the sheet surface parallel to the electric field:

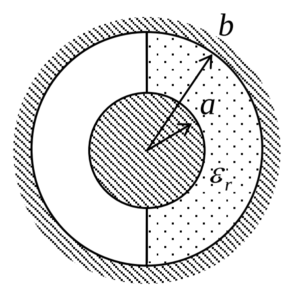
$$E = E_0, \quad D = \varepsilon E = \varepsilon_r \varepsilon_0 E_0, \quad P = \chi_e \varepsilon_0 E = (\varepsilon_r - 1)\varepsilon_0 E_0 .$$

(ii) the surface perpendicular to the field:

$$D = D_0 = \varepsilon_0 E_0, \quad E = D / \varepsilon_r \varepsilon_0 = E_0 / \varepsilon_r, \quad P = \chi_e \varepsilon_0 E = [(\varepsilon_r - 1) / \varepsilon_r]\varepsilon_0 E_0 .$$

We see that in the latter case all the fields are ε_r times weaker than in the former one.

Problem M.6 (100 points). A coaxial cable is half-filled with a dielectric – see Fig. on the right, which shows the cross-section of the system. Find the cable's capacitance (i.e. the mutual capacitance between its outer and inner conductors) per unit length.



Hint: The answer is pretty simple but you have to prove rather than just guess it.

Solution: Using the same logic as was applied in class to a partly-filled plane capacitor (see Fig. 3.8a and its discussion in the lecture notes), we may *guess* that the electrostatic potential is a function of only the distance from the system axis: $\Phi = \Phi(\rho)$. Now let us *prove* that this is *a* correct (and hence *the* correct) solution of our boundary problem:

(i) This distribution satisfies the Laplace equation, provided that the function $\Phi(\rho)$ has the form (following, e.g., from Eq. (2.103) of the lecture notes):

$$\Phi = A + B \ln \rho . \qquad (*)$$

(ii) The boundary conditions (of the continuity of Φ and $\varepsilon\partial\Phi/\partial n$) on the dielectric surface are satisfied – the second one because for our solution $\partial\Phi/\partial n = \partial\Phi/\partial\varphi = 0$.

(iii) The boundary conditions on each conductor surface (Φ = const) are also satisfied, provided that constants A and B in Eq. (*) are properly chosen:

$$\Phi_a = A + B \ln a, \quad \Phi_b = A + B \ln b ,$$

giving (for Φ_b - $\Phi_a \equiv V$):

$$B = \frac{V}{\ln(b/a)}, \quad \Phi = V\frac{\ln\rho}{\ln(b/a)} + \text{const}, \quad E = E_\rho = -\frac{V}{\rho\ln(b/a)}.$$

So the proof is complete, and we may use our solution to calculate the charge density on, say, external conductor:[2]

$$\sigma = D_n\Big|_{\rho=b} = -D_\rho\Big|_{\rho=b} = \frac{\varepsilon_0 V}{b\ln(b/a)} \times \begin{cases} 1, & \text{in the vacuum part,} \\ \varepsilon_r, & \text{in the dielectric part.} \end{cases}$$

The full charge (per unit length) is

$$\frac{Q}{L} = \pi b\varepsilon_0 \frac{V}{b\ln(b/a)}(\varepsilon_r + 1) = \frac{2\pi\varepsilon_0}{\ln(b/a)} V \frac{\varepsilon_r + 1}{2}.$$

As a result, the capacitance per unit length is

$$\frac{C_m}{L} = \frac{2\pi\varepsilon_0}{\ln(b/a)} \frac{\varepsilon_r + 1}{2}.$$

This is exactly the average between the values for the fully-vacuum and fully-dielectric-filled cables.

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[2] It may be readily checked (but is evident from the Gauss law) that the charge of the internal electrode is equal and opposite.