

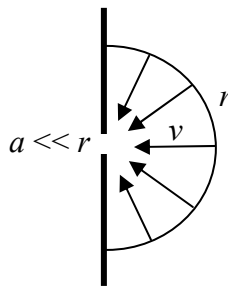
## Homework 03 with Solutions

**Problem 3.1.** An ideal classical gas, confined in a container with the linear size scale  $L$ , had been in thermal equilibrium at temperature  $T$ . Then a small hole of size  $a$  was opened in the wall of the container for a short time interval  $t$  such that  $a \ll v_0 t \ll L$ , where  $v_0$  is the r.m.s. velocity of the molecules in equilibrium:

$$v_0 \equiv \langle v^2 \rangle^{1/2} = \left( \frac{3T}{m} \right)^{1/2}.$$

Find the r.m.s. velocity of the escaped molecules. Compare it with  $v_0$ . On the basis of the comparison, what would be the most immediate observable effect of the gas emission?

*Solution:* Due to the condition  $v_0 t \ll L$ , during the period of the hole being open, we can neglect collisions of molecules with walls. Hence only the molecules flying directly into the hole may escape (see Fig. below).



Moreover, velocity of such molecules should satisfy the condition  $v \geq r/t$ , where  $r$  is the initial distance of the molecule from the hole. (Due to the condition  $a \ll r$  satisfied for most escaped molecules, it is not important which exactly part of the hole we are speaking about.) Hence, the number of escaped molecules with velocities within the range  $[v, v+dv]$  is proportional to  $r^3 = (vt)^3$ :

$$dn = \text{const} \times (vt)^3 \rho(v) dv,$$

where the distribution of initial velocities is described by the Maxwell formula,

$$\rho(v) \propto \exp\left\{-\frac{mv^2}{2T}\right\}.$$

(Since molecules from each point may reach the hole only if they fly in a certain direction, the distribution should be for one velocity component only.) Hence

$$dn = c(vt)^3 \exp\left\{-\frac{mv^2}{2T}\right\} dv,$$

(where  $c$  is some constant), and the average  $v^2$  of the escaped molecules can be calculated as

$$\langle v^2 \rangle = \frac{\int v^2 dn}{\int dn} = \frac{c \int_0^\infty v^2 (vt)^3 \exp\left\{-\frac{mv^2}{2T}\right\} dv}{c \int_0^\infty (vt)^3 \exp\left\{-\frac{mv^2}{2T}\right\} dv} = \frac{2T \int_0^\infty \xi^5 \exp\{-\xi^2\} d\xi}{m \int_0^\infty \xi^3 \exp\{-\xi^2\} d\xi} = \frac{2T}{m} \frac{1}{1/2} = \frac{4T}{m}.$$

(We have used the values of two table integrals.) Hence the r.m.s. velocity ratio is

$$\frac{\langle v^2 \rangle^{1/2}}{\langle v_0^2 \rangle^{1/2}} = \sqrt{\frac{4}{3}} \approx 1.154.$$

The immediately observable result of escape of the hotter molecules is cooling of the remaining gas.

**Problem 3.2.** A vessel with an ideal classical gas is separated by a partition so that the number  $N$  of molecules in both parts is the same but their volumes are different. After the system has reached thermal equilibrium, the gas pressure in one part is  $p_1$ , and in another,  $p_2$ . Calculate the change of entropy caused by a fast removal of the partition. Analyze the result.

*Solution:* Both in class (using the Gibbs distribution) and in Homework 2 (starting from the microcanonical distribution) we have obtained the following formula for the entropy of an ideal classical gas:

$$S = N \left[ \ln \frac{V}{N} + \varphi(T) \right],$$

where  $\varphi(T)$  is some function of temperature. Using the equation of state of an ideal gas,  $pV = NT$ , we may present this relation as

$$S = N \left[ \ln \frac{T}{p} + \varphi(T) \right],$$

so that the initial entropy of our system is

$$S_i = N \left[ \ln \frac{T}{p_1} + \varphi(T) \right] + N \left[ \ln \frac{T}{p_2} + \varphi(T) \right],$$

while its energy

$$E = 2 \frac{3}{2} NT.$$

Since during a fast removal of the partition the energy has no time to change, and the total number of molecules ( $2N$ ) is the same, temperature also cannot change.

(Whether it would change later on, depends on the coupling between the considered vessel and its environment. If the vessel is thermally insulated, neither  $E$  nor  $N$  would change.) Thus the entropy after the partition removal is

$$S_f = 2N \left[ \ln \frac{T}{p_f} + \varphi(T) \right],$$

where the final pressure  $p_f$  may be found from the evident condition  $V_f = V_1 + V_2$  and the equation of state applied to each of its terms:

$$\frac{(2N)T}{p_f} = \frac{N_1 T}{p_1} + \frac{N_2 T}{p_2}.$$

From here

$$\frac{1}{p_f} = \frac{1}{2} \left( \frac{1}{p_1} + \frac{1}{p_2} \right) = \frac{p_1 + p_2}{2p_1 p_2},$$

and the entropy change is

$$\Delta S \equiv S_f - S_i = 2N \left[ \ln \frac{T}{p_f} \right] - N \left[ \ln \frac{T}{p_1} \right] - N \left[ \ln \frac{T}{p_2} \right] = N \ln \frac{p_1 p_2}{p_f^2} = N \ln \frac{(p_1 + p_2)^2}{4p_1 p_2}.$$

The expression under the logarithm reaches its minimum (equal to one 1) if  $p_1 = p_2$ . In this case  $\Delta S = 0$ , while for any other relation of the initial pressures,  $\Delta S > 0$ , in accordance with the 2<sup>nd</sup> Law of thermodynamics.